A CHARACTERIZATION OF ALL ALGEBRAS IN THE VARIETY GENERATED BY AN IRREDUCIBLE MONO-UNARY ALGEBRA

สำนักหอสมุดกลาง

By
Mr.Sujin Sukgumpapan

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree
Master of Science Program in Mathematics
Department of Mathematics
Graduate School, Silpakorn University
Academic Year 2012
Copyright of Graduate School, Silpakorn University

A CHARACTERIZATION OF ALL ALGEBRAS IN THE VARIETY GENERATED BY AN IRREDUCIBLE MONO-UNARY ALGEBRA

สำนักหอสมุดกลาง
มหาวิทยาลัยศิลปากร

By
Mr.Sujin Sukgumpapan

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree
Master of Science Program in Mathematics
Department of Mathematics
Graduate School, Silpakorn University
Academic Year 2012
Copyright of Graduate School, Silpakorn University

ลักษณะเฉพาะของพีชคณิตในวาไรตีที่ก่อกำเนิดโดยพีชคณิตเอกภาคหนึ่งเดียวชนิดลดทอนไม่ได้

สำนักหอสมุดกลาง

โดย

นายสุจินต์ สุขกุมภาพันธ์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาคณิตศาสตร์

ภาควิชาคณิตศาสตร์

บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร

ปีการศึกษา 2555

ลิขสิทธิ์ของบัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร

The Graduate School, Silpakorn University has approved and accredited the Thesis title of "A characterization of all algebras in the variety generated by an irreducible algebra" submitted by MR.Sujin Sukgumpapan as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics

.....

(Assistant Professor Panjai Tantatsanawong,Ph.D.)

Dean of Graduate School

...../...../.....

The Thesis Advisor

Professor Chawewan Ratanaprasert, Ph.D.

The Thesis Examination Committee

..... Chairman

(Jitti Rukbud, Ph.D.)

...../...../.....

..... Member

(Associate Professor Pattanee Udomkavanich, Ph.D.)

...../...../.....

..... Member

(Professor Chawewan Ratanaprasert, Ph.D.)

...../...../.....

สำนักหอสมุดกลาง มหาวิทยาลัยศิลปากร

53305210 : MAJOR : MATHEMATICS
KEY WORD : SUBDIRECTLYIRREDUCIBLE/TERM/IDENTITY/VARIETY
SUJIN SUKGUMPAPAN : A CHARACTERIZATION OF ALL ALGEBRAS IN THE VARIETY GENERATED BY AN IRREDUCIBLE MONO-UNARY ALGEBRA.
THESIS ADVISOR : PROF.CHAWEWAN RATANAPRASERT, Ph.D. 31 pp.

Let $\bar{A}$ be a loop with non-branched tail and I be an index set. Then a product $\bar{A}^{I}$ is a circle of $2^{|I|}$-sheets. We characterize all algebras in the variety generated by $\bar{A}$ by describing all homomorphic images of subalgebras of a circle. If $\bar{A}$ is a cycle with finite length then $\bar{A}^{I}$ is $(|A|,0)$-bounded whose all connected components are a cycle with length $|A|$. We characterize all algebras in the variety generated by $\bar{A}$ by describing all homomorphic images of $\bar{A}$.

We also prove that for a positive integer n and nonnegative integer k, a variety of the form $\mathcal{A}(n,0)$ is generated by a cycle and a variety of the form $\mathcal{A}_c(1,k)$ is generated by a loop with non-branch tail.

Department of Mathematics Graduate School, Silpakorn University
Student's signature ..... Academic Year 2012
Thesis Advisor's signature .....

53305210 : สาขาวิชาคณิตศาสตร์
คำสำคัญ : พีชคณิตละทอนไม่ได้เชิงผลคูณย่อย/เทอม/เอกลักษณ์/วาไรตี

สุจินต์ สุขกุมภาพันธ์ : ลักษณะเฉพาะของพีชคณิตในวาไรตีที่ก่อกำเนิดโดยพีชคณิตเอกภาคหนึ่งเดียวชนิดลดทอนไม่ได้. อาจารย์ที่ปรึกษาวิทยานิพนธ์ : ศ.ดร.ฉวีวรรณ รัตนประเสริฐ. 31 หน้า.

ให้ $\bar{A}$ เป็น a loop with non-branched tail และ I เป็นเซตดรรชนีแล้วผลคูณตรง $\bar{A}^{I}$ ของ $\bar{A}$ เป็น a circle of $2^{|I|}$-sheets เราให้ลักษณะเฉพาะของพีชคณิตทั้งหมดในวาไรตีที่ก่อกำเนิดโดย $\bar{A}$ โดยการบรรยาย homomorphic images ทั้งหมดของพีชคณิตย่อยของ a circle ถ้า $\bar{A}$ เป็น cycle ที่มีขนาดจำกัดแล้วผลคูณตรง $\bar{A}^{I}$ เป็น $(|A|,0)$-bounded ซึ่งทุก ๆ connected components เป็น cycle ที่มีขนาด $|A|$ เราให้ลักษณะเฉพาะของพีชคณิตทั้งหมดในวาไรตีที่ก่อกำเนิดโดย $\bar{A}$ โดยการบรรยาย homomorphic images ทั้งหมดของ $\bar{A}$

เราได้พิสูจน์ด้วยว่าสำหรับแต่ละจำนนวนเต็มบวก n วาไรตีในรูป $\mathcal{A}(n, 0)$ เป็นวาไรตีที่ก่อกำเนิดโดย a cycle และสำหรับแต่ละจำนวนเต็มที่ไม่เป็นลบ k วาไรตีในรูป $\mathcal{A}_c(1, k)$ ก่อกำเนิดโดย a loop with non-branched tail

ภาควิชาคณิตศาสตร์ บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร
ลายมือชื่อนักศึกษา..... ปีการศึกษา 2555
ลายมือชื่ออาจารย์ที่ปรึกษาวิทยานิพนธ์ .....

Acknowledgments

This thesis has been completed by the involvement of people about whom I would to mention here.

I would like to thank Prof. Dr. Chawewan Ratanaprasert, my advisor for her valuable suggestions and excellent advices throughout the study with great attention.

I would like to thank Dr.Jitti Rukbud and Assoc. Prof. Pattanee Udomkavanich, Chairman and Member of the thesis Committee, for their valuable comments and suggestions.

Finally, I would like to express my gratitude to my family and my friends for their understanding, encouragement and moral support during the study.

สำนักหอสมุดกลาง
มหาวิทยาลัยศิลปากร

Table of Contents

	Page
Abstract in English	d
Abstract in Thai	e
Acknowledgments	f
List of Figures	h
Chapter	
1 Introduction	1
2 Basic concept	3
Algebras	3
Mono-unary algebras	7
Lattices	7
3 Classification of all varieties of mono-unary algebras	9
Classification of all varieties of mono-unary algebras	9
4 A characterization of all algebras in the variety generated by an irreducible algebra	19
A direct product of a loop with non-branched tail	19
A subalgebra of a circle	23
A homomorphic image of a circle	25
A variety generated by a loop with non-branched tail	26
A variety generated by a cycle	27
Bibliography	30
Biography	31

List of Figures

Figures		Page
1	A mono-unary algebra	10
2	A loop with branched tail	11
3	A mono-unary algebra which contains no cycle	11
4	A (12,3)-bounded algebra	13
5	A (1,8)-bounded algebra	13
6	A connected (1,12)-bounded algebra	14
7	A cycle with length 4	15
8	A circle of 2^2-sheets	21
9	A circle of 2^2-sheets with radius 4	21
10	A circle of 2^3-sheets with radius 2	22
11	A sheet of a circle of 2^2-sheets with radius 4	24
12	A subsheet of a circle of 2^2-sheets with radius 4	25
13	A subalgebra of a circle of 2^2-sheets with radius 4	25
14	A product $\overline{A}^2$ of a cycle $\overline{A}$ with length 4	28
15	A product $\overline{A}^3$ of a cycle $\overline{A}$ with length 3	28

Chapter 1

Introduction

Let A be a non-empty set and let I be a non-empty index set. Let $(f_i)_{i\in I}$ be a sequence which assigns to every element i of I is an n_i-ary operation f_i defined on A. Then the pair $\bar{A} = (A; (f_i)_{i\in I})$ is called an (indexed) **algebra**. The set A is called the **universe** of $\bar{A}$ and $(f_i)_{i\in I}$ is called the **sequence of fundamental operations** of $\bar{A}$. For each $i \in I$, the natural number n_i is called the **arity** of f_i. The sequence $\tau := (n_i)_{i\in I}$ of all arities is called the **type** of $\bar{A}$. We denote by $Alg(\tau)$, the class of all algebras of a given type τ. If $\tau = (1)$, $\bar{A}$ is called a **mono-unary algebra**. In this case, we denote (A, f) instead of $(A; (f_i)_{i\in I})$.

Let n be a natural number and $X_n = \{x_1, \ldots, x_n\}$ be an n-element set. The set X_n is called an **alphabet** and its elements are called **variables**. We also need a set $\{f_i | i \in I\}$ of operation symbols, indexed by a set I. The sets X_n and $\{f_i | i \in I\}$ are disjoint. To every operation symbol f_i, we assign a natural number $n_i \geq 1$ which is called the **arity** of f_i. As in the definition of an algebra, the sequence $\tau := (n_i)_{i\in I}$ of all arities is called the **type of the language**. The **n-ary terms** of type τ over X are defined in the following induction way: (i) every variable $x_i \in X_n$ is an n-ary term; and (ii) if $t_1, \ldots, t_{n_i}$ are an n-ary terms and f_i is an n_i-ary operation symbol then $f_1(t_1, \ldots, t_{n_i})$ is an n-ary term. The set $W_\tau(X_n)$ of all n-ary terms is the smallest set which contains $x_1, \ldots, x_n$ and is closed under finite application of (ii) above. We denote by $W\tau(X)$ the set of all terms of type τ over the countably infinite alphabet X.

Let $\bar{A}$ be an algebra of type τ and let t be an n-ary term of type τ over X. Then t induces an n-ary operation $t^{\bar{A}}$ on $\bar{A}$ called the **term operation** induced by the term t on the algebra $\bar{A}$ via the following steps: (i) if $t = x_i$ then $t^{\bar{A}}$ is an n-ary projection on A; and (ii) if $t = f_i(t_1, \ldots, t_{n_i})$ is n-ary term of type τ such that $t_1^{\bar{A}}, \ldots, t_{n_i}^{\bar{A}}$ are term operations which are induced by $t_1, \ldots, t_{n_i}$ then $t^{\bar{A}} = f^{\bar{A}}(t_1^{\bar{A}}, \ldots, t_{n_i}^{\bar{A}})$.

An **equation** of type τ is a pair of terms $(p, q) \in (W_\tau(X))^2$, and such pairs are more commonly written as $p \approx q$. An equation $p \approx q$ is said to be an **identity** in an algebra $\bar{A}$ of type τ if $p^{\bar{A}} = q^{\bar{A}}$; that is, if the term operations induced by p and q on $\bar{A}$ are equal. In this case, we also say that the equation $p \approx q$ is satisfied by $\bar{A}$, and we write $\bar{A} \models p \approx q$. The equation $p \approx q$ is said to be an identity in a class K of algebras of type τ if $p \approx q$ is an identity in $\bar{A}$ for all $\bar{A} \in K$, and we write $K \models p \approx q$.

Let $\sum$ be a set of equations of type τ and let K be a class of algebra

of type τ. We say that $\sum$ is **satisfied** by K (or $\bar{A}$) if $K \models p \approx q$ (or $\bar{A} \models p \approx q$) for all $p \approx q \in \sum$ and we write $K \models \sum$ (or $\bar{A} \models \sum$). In this case, we say that $\sum$ is the set of identities for K ($\bar{A}$ respectively).

We denote by $\mathscr{V} = Mod(\sum)$ the class of algebras of type τ which satisfy $\sum$ and $\mathscr{V}$ is called the **variety** of algebras type τ whose set of identities is $\sum$.

D.jakubikova -Studenovaska [1] described a variety of mono-unary algebras that is a model class of some set of identities $\sum$. She has classified all identities in $\sum$. There are exactly four forms of identities as follows:

$$f^k(x) \approx f^k(x); k \in \mathbb{N}_0$$

$$f^k(x) \approx f^k(y); k \in \mathbb{N}_0$$

$$f^{n+k}(x) \approx f^{n+k}(x); n \in \mathbb{N}, k \in \mathbb{N}_0$$

$$f^{n+k}(x) \approx f^{n+k}(y); n \in \mathbb{N}, k \in \mathbb{N}_0.$$

She has also classified all varieties of mono-unary algebras by identities. She observed that there are three forms of varieties of mono-unary algebras which are model classes of these identities, the class of all mono-unary $\mathscr{U}$ algebras, the classes of (n, k)-bounded algebras $\mathscr{A}(n, 0)$ and the classes of connected $(1, k)$-bounded algebras $\mathscr{A}_c(1, k)$ for all positive integers n and nonnegative integer k.

On the other hand, G. Birkhoff [2] proved that a variety $\mathscr{V}$ is the class of all homomorphic images of subalgebras of direct products of some classes K of algebras;that is , $\mathscr{V} = HSP(K)$ and he says that the variety $\mathscr{V}$ is generated by the class K. In the recent year, Y. Micheal [3] observed that there are two forms of subdirectly irreducible mono-unary algebras which are either loops with non-branched tail or cycles of prime power length.

In the thesis, we prove that for a positive integer n, a variety of the form $\mathscr{A}(n, 0)$ is generated by a cycle $\bar{A}$; and we characterize all algebras in this variety by describing all homomorphic images of $\bar{A}$.

We prove that for some nonnegative integer k, a variety of the form $\mathscr{A}_c(1, k)$ is generated by a loop with non-branched tail $\bar{A}$; and we characterize all algebras in this variety by describing all homomorphic images of subalgebras a circle.

Chapter 2

Basic Concepts

In this chapter, basic concepts related to the main results which are presented in chapter 4, are given.

สำนักหอสมุดกลาง

2.1 Algebra

Definition 2.1.1. Let A be a set. For a positive integer n, a function $f : A^n \to A$ is called an n**-ary operation** defined on A and is said to have **arity** n.

If $n = 1, 2, 3$, we call f a **unary operation**, **binary operation** and **ternary operation** respectively.

Definition 2.1.2. Let A be a nonempty set and F be a set of operations on A. Then a pair $\bar{A} = (A, F)$ is called an **algebra**. The set A is called the **universe** of $\bar{A}$. Every element in F is called a **fundamental operation** of $\bar{A}$.

If $F = \{f_i | i \in I\}$ is a function which assigns to every element of I an n_i-ary operation f_i defined on $\bar{A}$ then $\bar{A}$ is called an **index algebra**. The sequence $\tau := (n_i)_{i\in I}$ of all the arities is called **type** of the algebra $\bar{A}$. We use the name $Alg(\tau)$ for the class of all algebras of a given type τ. In the case that, F has only one element f which is unary operation we denote a pair (A, F) by (A, f) and it is called a **mono-unary algebra**.

Definition 2.1.3. Let $\bar{A} = (A, (f_i^A)_{i\in I})$ be an algebra of type τ. Then an algebra $\bar{B}$ is called a **subalgebra** of $\bar{A}$, written as $\bar{B} \subseteq \bar{A}$, if the following conditions are satisfied:
(i) $B \subseteq A$;
(ii) $\bar{B} = (B, (f_i^B)_{i\in I})$ is an algebra of type τ;
(iii) for all $i \in I$, f_i^B is $f_i^A|_B$.

Definition 2.1.4. Let $\bar{A} = (A, (f_i^A)_{i\in I})$ and $\bar{B} = (B, (f_i^B)_{i\in I})$ be algebras of the same type τ. Then a function $h : A \to B$ is called a **homomorphism** of $\bar{A}$ into $\bar{B}$ if for all $i \in I$ we have

$$h(f_i^A(a_1, \ldots, a_{n_i})) = f_i^B(h(a_1), \ldots, h(a_{n_i})),$$

for all $a_1, \ldots, a_{n_i} \in A$. If a homomorphism h is surjective h is called an **epimorphism**. A homomorphism h is called **isomorphism** if it is bijective. An injective homomorphism from $\bar{A}$ to $\bar{B}$ is also called an **embedding** of $\bar{A}$ in to $\bar{B}$.

Definition 2.1.5. Let $(\bar{A}_j)_{j\in J}$ be a family of algebras of type τ. The **direct product** $\prod\limits_{j\in J} A_i$ of the $\bar{A}_j$ is defined as an algebra with the carrier set

$$P := \prod_{j\in J} \bar{A}_j := \{(a_j)_{j\in J} | \forall j \in J(a_j \in A_j)\}$$

and the operations

$$f_i^P((a_{1j})_{j\in J}, \ldots, (a_{n_j j})_{j\in J}) = (f_i^A(a_{1j}, \ldots, a_{n_j j}))_{j\in J}.$$

If for all $j \in J$, $\bar{A}_i = \bar{A}$ then we usually write $\bar{A}^J$ instead of $\prod\limits_{j\in J} \bar{A}_j$. If $J = \phi$ then $\bar{A}^\phi$ is defined to be the one-element (trivial) algebra of type τ. If $J = \{1, \ldots, n\}$ then the direct product can be written as $\bar{A}_1 \times \ldots \times \bar{A}_n$.

Definition 2.1.6. An algebra $\bar{A}$ is said to be **directly irreducible** if whenever $\bar{A} \cong \bar{B}_1 \times \bar{B}_2$, either $|B_1| = 1$ or $|B_2| = 1$.

Definition 2.1.7. Let $(\bar{A}_j)_{j\in J}$ be a family of algebras of type τ. A subalgebra $\bar{B} \subseteq \prod\limits_{j\in J} \bar{A}_j$ of the direct product of $(\bar{A}_j)_{j\in J}$ is called a **subdirect product** of the algebra $\bar{A}_j$, if $h_k(\bar{B}) = \bar{A}_k$ for every projection mapping $h_k : \prod\limits_{j\in J} \bar{A}_j \to \bar{A}_k$. An embedding $\alpha : \bar{A} \to \prod\limits_{j\in J} \bar{A}_j$ is **subdirect** if $\alpha(\bar{A})$ is a subdirect product of the $(\bar{A}_j)_{j\in J}$.

Definition 2.1.8. An algebra $\bar{A}$ is **subditectly irreducible** if for every subdirect embedding $\alpha : \bar{A} \to \prod\limits_{j\in J} \bar{A}_j$ there is an $j \in J$ such that $\pi_j \circ \alpha : \bar{A} \to \bar{A}_j$ is an isomorhpism, where $\pi_j : \prod\limits_{j\in J} \bar{A}_j \to \bar{A}_j$ is a projection .

Definition 2.1.9. We define the following operators on the class $Alg(\tau)$ of all algebra of fixed type τ. For any class $K \subseteq Alg(\tau)$,
$S(K)$ is the class of all subalgebras of algebras from K,
$H(K)$ is the class of all homomorphic images of algebras from K,
$P(K)$ is the class of all direct products of families of algebras from K.

Definition 2.1.10. A subclass K of $Alg(\tau)$ is called a **variety** if K is closed under the operators H, S and P; that is, if $H(K) \subseteq K$, $S(K) \subseteq K$ and $P(K) \subseteq K$.

In general, we know that a group G is an algebra of type $\tau = (2)$. The association law $x \cdot (y \cdot z) \approx (x \cdot y) \cdot z$ is satisfied. Satisfaction of this law means that for all element $x, y, z \in G$, the equation $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ holds. To write this equation, we need the symbols x, y and z. But these symbols are not themselves elements of G; they are only symbols for which elements from G may be substituted. Such symbols are called variables. To write identities or laws of an

algebra we need a language, which must include such variables as well as symbols to represent the operations. In the associative law above we did not distinguish between the operation and the symbol used to denote it, using $\cdot$ for both , but in our new formal language we shall usually make such a distinction. That is , we will have formal operation symbols distinct from concrete operations on a set.

Now, we proceed to define this formal language in a general setting. Let $n \geq 1$ be a natural number. Let $X_n = \{x_1, \ldots, x_n\}$ be an n-element set. The set X_n is called an **alphabet** and its elements are called **variables**. We also need a set $\{f_i | i \in I\}$ of **operation symbols**, indexed by a set I. The sets X_n and $\{f_i | i \in I\}$ are disjoint. To every operation symbol f_i we assign a natural number $n_i \geq 1$, called the **arity** of f_i. As in the definition of an algebra, the sequence $\tau = (n_i)_{i \in I}$ of all the arities is called the **type** of $\{f_i | i \in I\}$.

Definition 2.1.11. For a natural number n, the **n-ary term** of type τ are defined in the following inductive way:
(i) every variable $x_i \in X_n$ is an n-ary term,
(ii) if $t_1, \ldots, t_{n_i}$ are n-ary term and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term.

The set $W_\tau(X_n) = W(x_1, \ldots, x_n)$ of all n-ary terms is the smallest set which contains $x_1, \ldots, x_n$ and is closed under finite application of (ii) in Definition 2.1.11. The set $W_\tau(X) = \bigcup_{n=1}^{\infty} W_\tau(X_n)$ denote the set of all terms of type τ over X.

Example 2.1.12. *1. If $\{f_i | i \in I\} = \{\cdot\}$ is the singleton set of a symbol of binary operation and $X = \{x, y, z\}$ then $x, y, z, xy, xz, x(xx), x(yx), x(yz), (xy)z$ are 3-ary terms of type (2)*
2. If $\{f_i | i \in I\} = \{\cdot, +\}$ is type $(2, 2)$ and $X = \{x, y, z\}$ then $x, y, z, xy, yz, x(y + z), (xy) + (xz)$ are 3-ary terms of type $(2, 2)$.

Terms are formal expression on our formal language of type τ. In oder to formulate statements using terms which are true or false in a given algebra $\bar{A}$, we have to evaluate the variables in the terms by elements of the concrete set A, and we have to interpret the operation symbols by concrete operations on this set. It is this process which produces term operations from terms. We shall continue to denote by X the countably infinite set $\{x_1, x_2, x_3, \ldots\}$ of variables.

Definition 2.1.13. Let $\bar{A}$ be an algebra of type τ and let t be an n-ary term of type τ over X. Then t induces an n-ary operation $t^{\bar{A}}$ on $\bar{A}$, called the **term operation** induced by the term t on the algebra $\bar{A}$, via the following steps:
(i) If $t = x_i \in \{x_1, \ldots, x_n\}$ then $t^{\bar{A}}$ is defined by $t^{\bar{A}}(a_1, \ldots, a_n) = a_i$ for all $a_1, \ldots, a_n \in A$.
(ii) If t is the form $f_i(t_1, \ldots, t_{n_i})$ where are $f_i \in \{f_i | i \in I\}$ and $t_1, \ldots, t_{n_i}$ are terms in $W_\tau(X_)$ such that $t_1^{\bar{A}}, \ldots, t_{n_i}^{\bar{A}}$ are defined, we let $t^{\bar{A}} := f_(^{\bar{A}} t_1^{\bar{A}}, \ldots, t_{n_i}^{\bar{A}})$ for all $a_1, \ldots, a_n \in A$ where $t^{\bar{A}}(a_1, \ldots, a_n) = f_i^{\bar{A}}(t_1^{\bar{A}}(a_1, \ldots, a_n), \ldots, t_{n_i}^{\bar{A}}(a_1, \ldots, a_n))$

Definition 2.1.14. For a natural number n, a pair $(p, q) \in (W_\tau(X))^2$ of arity n of type τ is called an **equation** written as $p \approx q$. Such an equation $p \approx q$ is said to be an **identity** in algebra $\bar{A}$ of type τ if $p^{\bar{A}} = q^{\bar{A}}$, that is, if the term operations

induced by p and q on the algebra $\bar{A}$ are equal. In this case we also say that the equation $p \approx q$ is satisfied or modelled by the algebra $\bar{A}$, and we write $\bar{A} \models p \approx q$. An equation $p \approx q$ is said to be an identity in class of algebras K of type τ if $p \approx q$ is identity in $\bar{A}$ for all $\bar{A} \in K$ and we write $K \models p \approx q$.

Example 2.1.15. *Let $\bar{A} = (A, f^{\bar{A}})$ be an algebra type $\tau = (2)$. We say that barA satisfies association law that is, the equation $x_1(x_2x_3) \approx (x_1x_2)x_3$ is the identity in $\bar{A}$;i.e.,*

$$\bar{A} \models x_1(x_2x_3) \approx (x_1x_2)x_3.$$

If f is a binary operation symbol then the equation

$$f(x_1, f(x_2, x_3)) \approx f(f(x_1, x_2), x_3)$$

arity $n \geq 3$ is the identity in $\bar{A}$; that is, we have

$$(f(x_1, f(x_2, x_3)))^{\bar{A}}(a, b, c) = (f(f(x_1, x_2), x_3))^{\bar{A}}(a, b, c),$$

where

$$\begin{aligned}(f(x_1, f(x_2, x_3)))^{\bar{A}}(a, b, c) &= f^{\bar{A}}(x_1(a, b, c), f^{\bar{A}}(x_2(a, b, c), x_3(a, b, c))) \\ &= f^{\bar{A}}(a, f^{\bar{A}}(b, c)\end{aligned}$$

and

$$\begin{aligned}(f((x_1, x_2), x_3))^{\bar{A}}(a, b, c) &= f^{\bar{A}}(f^{\bar{A}}(x_1(a, b, c), x_2(a, b, c)), x_3(a, b, c))). \\ &= f^{\bar{A}}(f^{\bar{A}}(a, b), c)\end{aligned}$$

Therefore, $f^{\bar{A}}(a, f^{\bar{A}}(b, c) = f^{\bar{A}}(f^{\bar{A}}(a, b), c)$. If we denote $f^{\bar{A}}$ by $\cdot$, we have

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

Definition 2.1.16. Let $\sum$ be a set of equations of type τ and let K be a class of algebras type τ. Then we say that $\sum$ satisfied by K (or $\bar{A}$) if $K \models p \approx q$ (or $\bar{A} \models p \approx q$) for all $p \approx q \in \sum$ and we write $K \models \sum$ (or $\bar{A} \models \sum$). In this case we say that, $\sum$ is the set of identities for K (or $\bar{A}$ respectively).

Definition 2.1.17. Let $\sum \subseteq (W_\tau(X))^2$ and $Mod(\sum) := \{\bar{A} \in Alg(\tau) | \bar{A} \models \sum\}$. We call $Mod(\sum)$ the variety of algebras such that $\sum$ is the set of identities for it.

2.2 Mono-unary Algebra

Definition 2.2.1. Let (A, f) be a mono-unary algebra. A mono-unary algebra $\bar{B} = (B, g)$ is called a **subalgebra** of (A,f) if $\phi \neq B \subseteq A$ and $f|_B = g$.

Definition 2.2.2. Let (A, f) and (B, g) be mono-unary algebras. A function $h : A \to B$ is called a homomorphism of A into B if $h(f(a)) = g(h(a))$ for all $a \in A$. If h is surjective then the homomorphism h is called an epimorphism from A onto B; and h is call an isomorphism from A to B if h is bijective.

2.3 Lattice

Definition 2.3.1. A binary relation r on a set A is called an **order** on A if the following three conditions hold for all $a, b, c \in A$:
(i) ara, (reflexivity)
(ii) arb and bra implies $a = b$, (antisymmetry)
(iii) arb and brc imply arc. (transitivity)

Definition 2.3.2. Let A be a set and r be an order on A. A pair $\bar{A} = (A, r)$ is called an **order set**.

Definition 2.3.3. Let r be a relation on a set A. It is said to be an **equivalent relation** if it is reflexive , transitive and symmetric.

Definition 2.3.4. Let r be an equivalent relation on A. For $a \in A$, the set $[a] = \{x \in A | xra\}$ is called an **equivalence class** of a relative to r and we denote by $A/r = \{[a] | a \in A\}$ the set of all equivalence classes relative to r.

Definition 2.3.5. Let $(\bar{A}, \leq)$ be an order set. An element $a \in A$ is called a **greatest element** of A if $c \leq a$ for all $c \in A$. An element $b \in A$ is called a **least element** of A if $b \leq c$ for all $c \in A$.

Definition 2.3.6. Let $(A, \leq)$ be an order set and $S \subseteq A$. An element x in A is called an **upper bound** of S if $s \leq x$ for all $s \in S$. An element y in A is called **lower bound** if $x \leq s$ for all $s \in S$.

We denote S^u and S^l the set of all upper bounds of S and the set of all lower bounds of S respectively. Then we have

$$S^u = \{x \in A | (\forall s \in S)(s \leq x)\} \text{ and } S^l = \{x \in A | (\forall s \in S)(x \leq s)\}.$$

If S^u has the least element x then x is called the **least upper bound** of S. If S^l has the greatest element y then y is called the **greatest lower bound** of S.

Definition 2.3.7. Let $\bar{A}$ be an order set. $\bar{A}$ is called a **lattice** if the greatest lower bound and the least upper bound of the set $\{a, b\}$ are in A for all $a, b \in A$.

Definition 2.3.8. Let $\bar{A} = (A, r)$ be an order set. $\bar{A}$ is called a **chain** if a and b are comparable for all $a, b \in A$; that is, arb or bra for all $a, b \in A$.

Let $\bar{L}$ be a lattice. Since every pair of elements in L has the unique greatest lower bound and the least upper bound, we denote by $a \vee b$ the least upper bound of the set {a,b} and $a \wedge b$ the greatest lower bound of the set {a,b}. We consider "$\vee$" and "$\wedge$" as binary operations in $\bar{L}$; that is, every lattice is an algebra which "$\vee$" and "$\wedge$" are its operations.

Definition 2.3.9. Let $\bar{L}$ be a lattice and $\phi \neq M \subseteq L$.Then M is a **sublattice** of $\bar{L}$ if $x \wedge y$ and $x \vee y$ are in M for all $x, y \in M$.

Definition 2.3.10. Let $\bar{L}$ and $\bar{K}$ be lattices and $f : L \to K$ be a function. Then:
(i) f is a homomorphism if $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in L$,
(ii) f is an isomorphism if f is a homomorphism and f is bijective.

Theorem 2.3.11. *Let $\bar{L} = (L; \leq_1)$ and $\bar{K} = (K; \leq_2)$ be order sets such that $L \cap K = \phi$. We define an order "R" on $L \cup K$ as follow: for all $a, b \in L \cup K$, aRb if and only if ($a, b \in L$ and $a \leq_1 b$) or ($a, b \in K$ and $a \leq_2 b$) or ($a \in L$ and $b \in K$). Then $(L \cup K; R)$ is an order set.*

The order set $(L \cup K; R)$ in Theorem 2.3.11 is called the **linear sum** of $\bar{L}$ and $\bar{K}$ and we denote it by the symbol $\bar{L} \oplus \bar{K}$. If $\bar{L}$ has exactly one element, we denote $\bar{L} \oplus \bar{K}$ by $1 \oplus \bar{K}$. If $\bar{K}$ has exactly one element, we we denote $\bar{L} \oplus \bar{K}$ by $\bar{L} \oplus 1$.

Theorem 2.3.12. *Let $\bar{L}$ and $\bar{K}$ be lattices. Then $\bar{L} \oplus \bar{K}$ is a lattice.*

Let $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of all positive integers and the set of all nonnegative integers respectively.

Let $\overline{\mathbb{N}} = (\mathbb{N}; \leq)$ and $\overline{\mathbb{N}}_0 = (\mathbb{N}; \leq)$ where "$\leq$" is the general order on $\mathbb{N}$.

Let $\overline{\mathbb{D}} = (\mathbb{N}; |)$ where "$|$" be defined by $a|b$ if and only if there exists $n \in \mathbb{N}$ such that $b = na$ for all $a, b \in \mathbb{N}$. $\overline{\mathbb{D}}_0 = (\mathbb{D} \cup \{0\}; r)$ where "r" denote the order which is defined as follow: arb if $a|b$ for all $a, b \in \mathbb{D}$ and $a|0$ for all $a \in \mathbb{D}_0$.

Theorem 2.3.13. *The following statements are true;*
(i) *$\overline{\mathbb{N}}$ and $\overline{\mathbb{N}}_0$ are chains,*
(ii) *$\overline{\mathbb{D}}$ and $\overline{\mathbb{D}}_0$ are lattices.*

Definition 2.3.14. Let $\bar{L}$ be a lattice. Then $\bar{L}$ is said to be a **distributive** lattice if $\bar{L}$ satisfies distributive law; that is, $\bar{L} \models x \vee (y \wedge z) \approx (x \wedge y) \vee (x \wedge z)$.

Example 2.3.15. *$\overline{\mathbb{N}}, \overline{\mathbb{N}}_0, \overline{\mathbb{D}}$ and $\overline{\mathbb{N}}_0$ are distributive lattices.*

Theorem 2.3.16. *Every sublattice of a distributive lattice is a distributive lattice*

Theorem 2.3.17. *Let $\bar{L}$ and $\bar{K}$ be distributive lattices. Then $\bar{L} \times \bar{K}$ is a distributive lattice.*

Theorem 2.3.18. *Let $\bar{L}$ be a distributive lattice. Then $\bar{L} \oplus 1$ and $1 \oplus \bar{L}$ are distributive lattices.*

Theorem 2.3.19. *Let $\mathscr{L}$ be the class of all varieties of algebra type τ. Then $\overline{\mathscr{L}} = (\mathscr{L}; \subseteq)$ is a lattice such that $Alg(\tau)$ is the greatest element and algebras type τ whose singleton universe are the least elements.*

Chapter 3

Classification of all varieties of mono-unary algebras

3.1 Classification of all varieties of mono-unary algebras

D.jakubikova -Studenovaska [1] described a variety of mono-unary algebras; that is, a model class of some set of identities. All theorem are stated and proved by D.jakubikova -Studenovaska [1].

Let $\bar{A} = (A; f)$ be a mono-unary algebra. For each $a \in A$, we define $f^0(a) = a$ and $f^n(a) = f(f^{n-1}(a))$ for all $n \in \mathbb{N}$; and for each $b \in Imf$, we define $f^{-1}(b) = \{a \in A | f(a) = b\}$. If $a, b \in A$ with $f(a) = b$, we write $a \to b$.

Theorem 3.1.1. *Let $\bar{A} = (A; f)$ be a mono-unary algebra. Then the relation $\sim$ be defined on A by $a \sim b$ if and only if there exist $m, n \in \mathbb{N}_0$ such that $f^m(a) = f^n(b)$ for each $a, b \in A$ is an equivalence relation.*

Proof. Clearly, $\sim$ is reflexive and symmetric. Let $a, b, c \in A$ such that $a \sim b$ and $b \sim c$. Then there exist $m, n, p, q \in \mathbb{N}_0$ such that $f^m(a) = f^n(b)$ and $f^p(b) = f^q(c)$ which imply that $f^{m+p}(a = f^{n+p}(b) = f^{m+q}(c)$. Therefore, $a \sim b$; so, the relation $\sim$ is transitive. □

Definition 3.1.2. Let $\bar{A}$ be a mono-unary algebra and $\sim$ be a relation on A defined as in Theorem 3.1.1. Each equivalence class in the set $A/\sim$ is called a **connected component** of $\bar{A}$. $\bar{A}$ is **connected** if $A/\sim$ contains exactly one element which is $\bar{A}$.

Definition 3.1.3. Let $(A; f)$ be a mono-unary algebra. An element $c \in A$ is called a **cyclic** if $f^k(c) = c$ for some positive integer k. The set of all cyclic elements of some connected component of $(A; f)$ is called a **cycle** of $(A; f)$. The **length** of a cycle is its cardinal. A cycle with one element is called a **loop.**

Definition 3.1.4. A connected mono-unary algebra $(A; f)$ is said to be a **cycle** C **with a tail** T, if C is a cycle, $A = C \cup T$, $C \cap T = \phi$ and there is $c \in C$ such that $f(x) \in T \cup \{c\}$ for every $x \in T$.

Definition 3.1.5. Let $(A; f)$ be a cycle C with a tail T. Then T is called **branched** if $|f^{-1}(t) \setminus C| \leq 2$ for each $t \in T \cup \{c\}$, and there is a unique $b \in T \cup \{c\}$ with $|f^{-1}(b) \setminus C| = 2$. If $b \neq c$, then T is **properly branched**. If T is branched, then it will be called **shortly branched** whenever $f^{-1}(x) \setminus C = \phi$ for $x \in f^{-1}(b)$. In the case when $|f^{-1}(t) \setminus C| \leq 1$ for each $t \in T \cup \{c\}$, the tail T is **non-branched**. If $T = \phi$, we call that (A, f) is a cycle.

If (A, f) is a loop with non-branched tail whose $|A|$ is finite, (A, f) is a loop with non-branched tail with length $|A|$.

Example 3.1.6. *Let $A = \{0, 1, 2, \ldots, 12\}$ and let $f : A \to A$ be defined as follows:* $12 \to 11 \to 10 \to 9 \to 10$, $8 \to 7 \to 6 \to 5 \to 4 \to 3 \to 6$. $2 \to 1 \to 0 \to 0$. *Then $\bar{A}/\sim = \{\{0, 1, 2\}, \{3, 4, 5, 6, 7, 8\}, \{9, 10, 11, 12\}\}$ where $\{0\}$, $\{3, 4, 5, 6\}$, and $\{9, 10\}$ are cycles and we have $0, 3, 6, 9$ and 10 are cyclics.*

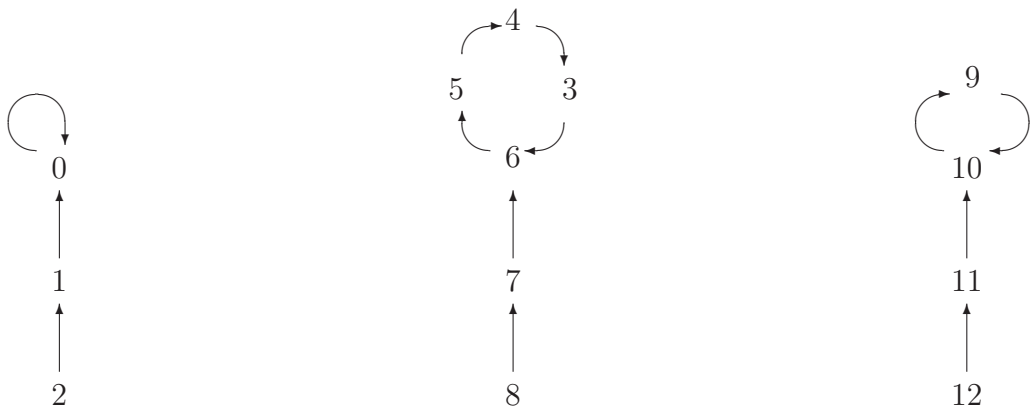

Figure 1. A mono-unary algebra which contain cycles with length $1, 4, 2$ respectively.

Example 3.1.7. *Let $A = \{0, 1, 2, \ldots, 14\}$ and let $f : A \to A$ be defined as follows:* $5 \to 4 \to 3 \to 2 \to 1 \to 0 \to 0$, $8 \to 7 \to 6 \to 1$, $10 \to 9 \to 6$, $13 \to 12 \to 3$, $11 \to 2$, $14 \to 1$, $15 \to 0$,. *Then $\bar{A}/\sim = \{A\}$ which shows that $\bar{A}$ is connected.*

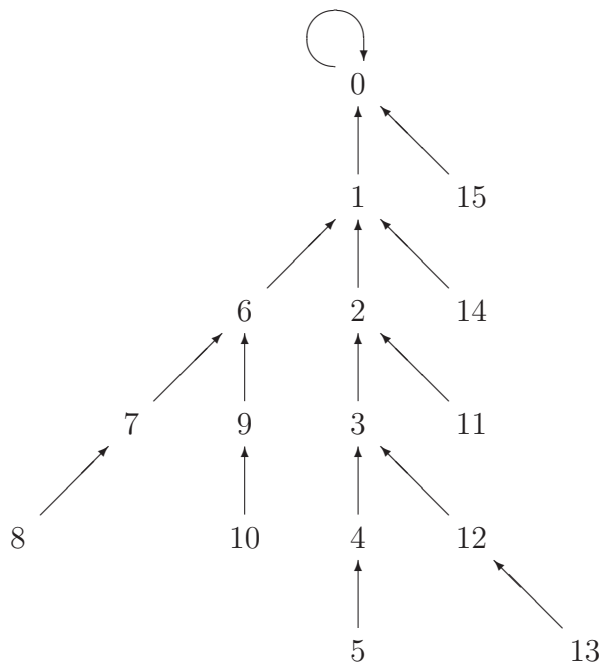

Figure 2. A loop with branched tail.

Example 3.1.8. *Let $A = \{0, 1, 2, \ldots\}$ and let $f : A \to A$ be defined as follow: $f(n) = n + 1$ for all $n \in \mathbb{N}_0$. Then $\bar{A}$ has no cycle.*

Figure 3. A mono-unary algebra which contains no cycle.

By previous examples we note that a connected component of a mono-unary algebra contains at least one cycle. By the definition of a cycle, if a component B contains a cycle C then C is the unique cycle in B. We have the following as a corollary of Theorem 3.1.1.

Corollary 3.1.9. *Let $\bar{A}$ be a mono-unary algebra and let $c \in A$. If there exists a positive integer k such that $f^k(c) = c$ then $f^n(c) \in \{c, f(c), \ldots, f^{k-1}(c)\}$ and $f^{nk}(c) = c$ for all $n \in \mathbb{N}$.*

Theorem 3.1.10. *Let $\bar{A}$ be a mono-unary algebra and B be a connected component in $\bar{A}$ which contains C as its cycle. If $a \in C$ then there exists $k \in \mathbb{N}$ such that $C = \{a, f(a), \ldots, f^{k-1}(a)\}$.*

Proof. Let $a \in C$. Then a is a cyclic and there exists $k \in \mathbb{N}$ such that $f^k(a) = a$; so, $\{a, f(a), \ldots, f^{k-1}\} \subseteq C$. Let $b \in C$. Then b is cyclic and there exists $n \in \mathbb{N}$ such that $f^n(b) = b$. Since $a, b \in C \subseteq B$, we have $a \sim b$ which implies that there exist $p, q \in \mathbb{N}_0$ such that $f^p(a) = f^q(b)$. Thus, $f^q(b) \in \{b, f(b), \ldots, f^{n-1}(b)\}$; so, there exists $0 \leq s < n$ such that $f^q(b) = f^s(b)$. Since $n - s \in \mathbb{N}$, we have $b = f^n(b) = f^{n-s+s}(b) = f^{n-s}(f^s(b)) = f^{n-s}(f^p(a)) = f^{n-s+p}(a)$. Hence, $b \in \{a, f(a), \ldots, f^{k-1}(a)\}$ which shows that $C \subseteq \{a, f(a), \ldots, f^{k-1}(a)\}$. $\square$

Remark 3.1.11. *Let $a \in A$. If there exists a positive integer m such that $f^m(a) = a$ then there exists the least positive integer k such that $f^k(a) = a$ and we have*
(i) $a, f(a), \ldots, f^{k-1}(a)$ are all distinct elements in C. So, $|C| = k$.
(ii) If $a \in A$ and there exists a positive integer m such that $f^m(a) = a$, the least positive integer k such that $f^k(a) = a$ is a factor of m.

In this chapter, we would like to describe the lattice of all varieties of mono-unary algebras by identities. We now look at all identities of mono-unary algebras. Since a mono-unary algebra is an algebra with the only one operation, every term in $W_\tau(X)$ is of the form $f^k(x)$ where $k \in \mathbb{N}_0$. So, there are exactly four forms of identities as stated in the following theorem.

Theorem 3.1.12. *Let $\bar{A}$ be a mono-unary algebra and f is an operation symbol on A. Then there exactly four forms of identities as follows:*

$$f^k(x) \approx f^k(x) \text{ where } k \in \mathbb{N}_0 \qquad (\alpha_k)$$

$$f^k(x) \approx f^k(y) \text{ where } k \in \mathbb{N}_0 \qquad (\beta_k)$$

$$f^{n+k}(x) \approx f^k(x) \text{ where } n \in \mathbb{N} \text{ and } k \in \mathbb{N}_0 \qquad (\gamma_{n,k})$$

$$f^{n+k}(x) \approx f^k(y) \text{ where } n \in \mathbb{N} \text{ and } k \in \mathbb{N}_0 \qquad (\delta_{n,k})$$

Proof. Let $p \approx q \in (W_\tau(X))^2$. Since f is the only operation on A, there are $m, k \in \mathbb{N}_0$ and $x, y \in X$ such that $p = f^m(x)$ and $q = f^k(y)$. If $m = k$ and $x = y$ then $p \approx q$ is of the form α_k. If $m = k$ and $x \neq y$ then $p \approx q$ is in the form β_k. Now, we consider the case $m \neq k$ and without loss of generality we may assume that $m > k$, there exists $n \in \mathbb{N}$ such that $m = n + k$. If $x = y$ then $p \approx q$ is in the form $\gamma_{n,k}$, and if $x \neq y$ then $p \approx q$ is in the form $\delta_{n,k}$. □

Definition 3.1.13. Let $\bar{A}$ be a mono-unary algebra. For each $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, $\bar{A}$ is called an (n, k)-bounded if for each component $B \in A/\sim$ the following conditions are satisfied:
(i) B contains the cycle C_B,
(ii) $|C_B|$ divides n,
(iii) $f^k(a) \in C_B$ for all $a \in B$.

Remark 3.1.14. *Every cycle $\bar{A}$ with finite length is connected $(|A|, 0)$-bounded.*

Remark 3.1.15. *If $\bar{A}$ is (n, k)-bounded for some positive integer n and nonnegative integer k then $\bar{A}$ is an (m, k)-bounded for all $m \in \mathbb{N}$ and $p \in \mathbb{N}_0$ such that $n|m$ and $k \leq p$.*

Example 3.1.16. *Let $A = \{0, 1, 2, \ldots, 12\}$ and let $f : A \to A$ be defined as follows:*
$12 \to 11 \to 10 \to 9 \to 8 \to 7 \to 9$, $6 \to 5 \to 4 \to 3 \to 6$, $2 \to 1 \to 0 \to 0$.
Then $A/\sim = \{\{0, 1, 2\}, \{3, 4, 5, 6\}, \{7, 8, 9, 10, 11, 12\}\}$ with $\{0\}$, $\{3, 4, 5, 6\}$ and $\{7, 8, 9\}$ are cycles with length $1, 4, 3$ respectively. Note that, $f^2(x) \in \{0\}$ for all $x \in \{0, 1, 2\}$, $f^0(x) \in \{3, 4, 5, 6\}$ for all $x \in \{3, 4, 5, 6\}$, $f^3(x) \in \{7, 8, 9\}$ for all $x \in \{7, 8, 9, 10, 11, 12\}$. Then $\bar{A}$ is a $(12, 3)$-bounded; and Remark 3.1.15 implies that $\bar{A}$ is a $(12n, k + 3)$-bounded for all $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$.

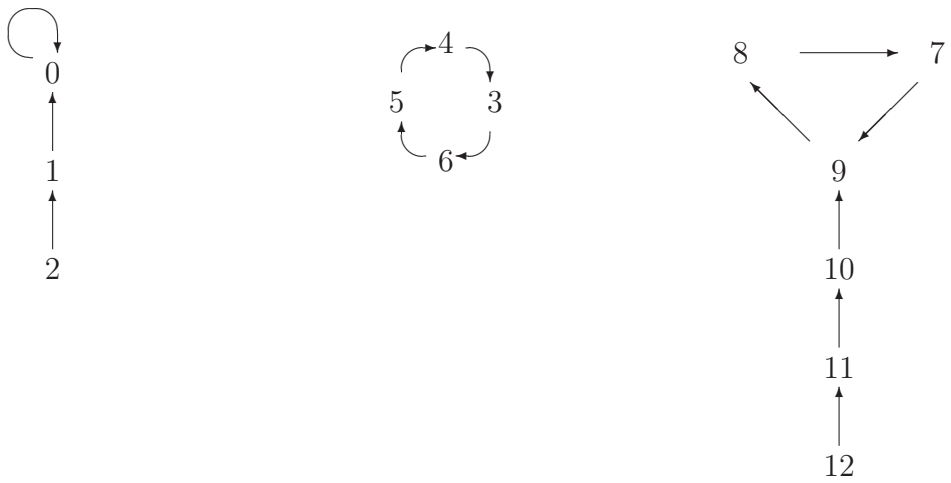

Figure 4. *A* $(12, 3)$*-bounded algebra.*

Example 3.1.17. *Let* $A = \{0, 1, 2, \ldots, 10\}$ *and let* $f : A \to A$ *be defined as follows:*
$10 \to 9 \to 9$, $8 \to 7 \to 6 \to 5 \to 4 \to 3 \to 2 \to 1 \to 0 \to 0$. *Then* $A/\sim = \{\{10, 9\}, \{0, 1, 2, 3, 4, 5, 6, 7, 8\}\}$ *with* $\{0\}$ *and* $\{9\}$ *are loops. Then* $\bar{A}$ *is an* $(1, 8)$*-bounded.*

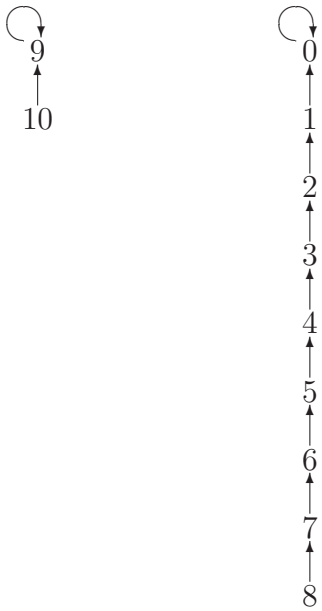

Figure 5. *A* $(1, 8)$*-bounded algebra*

Example 3.1.18. *Let* $A = \{0, 1, 2, \ldots, 15\}$ *and let* $f : A \to A$ *be defined as follows:*
$12 \to 11 \to 10 \to 9 \to 8 \to 7 \to 6 \to 5 \to 4 \to 3 \to 2 \to 1 \to 0 \to 0$, $14 \to 13 \to 5$ *and* $20 \to 19 \to 18 \to 17 \to 16 \to 15 \to 2$. *Then* $A/\sim = \{A\}$ *with* $\{0\}$ *is a loops. Then* $\bar{A}$ *is a connected* $(1, 12)$*-bounded.*

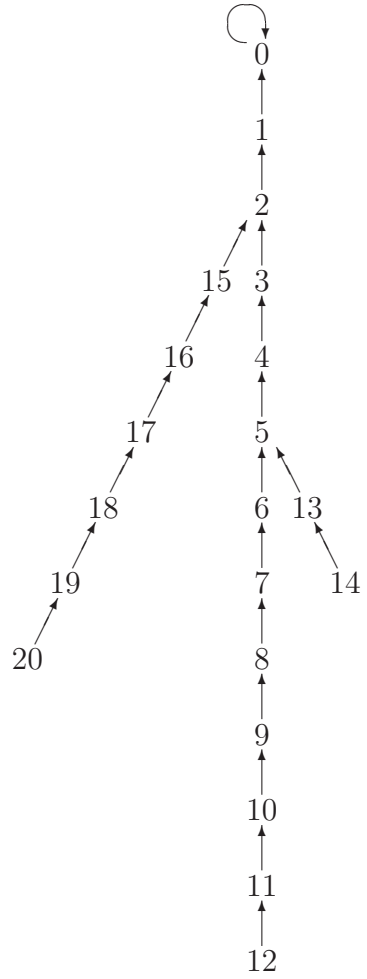

Figure 6. *A connected* $(1, 12)$*-bounded algebra*

Example 3.1.19. *Let* $A = \{0, 1, 2, 3, 4\}$ *and let* $f : A \to A$ *be defined as follows:* $4 \to 3 \to 2 \to 1 \to 4$ *Then* $\bar{A}$ *is a cycle with length* 4.

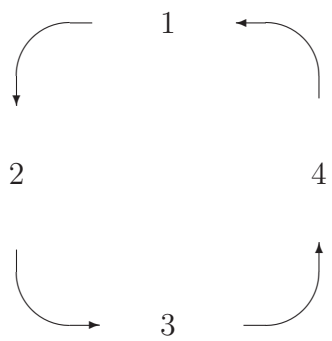

Figure 7. *A cycle with length* 4

Theorem 3.1.20. *Let* $\bar{A}$ *be a mono-unary algebra and* $k \in \mathbb{N}_0$. *Then* $\bar{A} \models f^k(x) \approx f^k(y)$ *if and only if* $\bar{A}$ *is connected* $(1, k)$*-bounded.*

Proof. Let $\bar{A}$ be connected $(1,k)$-bounded for some $k \in \mathbb{N}_0$. Then A is the only component in $\bar{A}$ which contains a cycle C with length 1. Let $a, b \in A$ then $f^k(a) \in C$ and $f^k(b) \in C$. Since $|C| = 1$, we have $f^k(a) = f^k(b)$ which shows that $\bar{A} \models f^k(x) \approx f^k(y)$. Conversely, let $a, b \in A$. Then $f^k(a) = f^k(b)$; so, $a \sim b$ which implies that A is the component in $A/\sim$. Therefore, A is connected. Let $a \in A$. Then $b = f(a) \in A$. By the assumption, $f^k(a) = f^k(b) = f^k(f(a)) = f^{k+1}(a) = f(f^k(a))$ which shows that $\{f^k(a)\}$ is a loop in $\bar{A}$. Then $\bar{A}$ is connected $(1,k)$-bounded. □

Theorem 3.1.21. *Let $\bar{A}$ be a mono-unary algebra. For $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, $\bar{A} \models f^{n+k}(x) \approx f^k(x)$ if and only if $\bar{A}$ is (n,k)-bounded.*

Proof. Let $\bar{A}$ be (n,k)-bounded and $a \in A$. Then there exists a connected component B of $\bar{A}$ and a cycle C in B such that $f^(a) \in C$ and $|C|$ devises n; that is, there exists an integer p such that $n = |C|p$. By Remark 3.1.15, we have $f^{n+k}(a) = f^n(f^k(a)) = f^{|C|p}(f^k(a)) = f^k(a)$. Therefore, $\bar{A} \models f^{n+k}(x) \approx f^k(x)$. □

Theorem 3.1.22. *Let $\bar{A}$ be a mono-unary algebra and $k \in \mathbb{N}$. Then $\bar{A} \models f^{n+k}(x) \approx f^k(y)$ if and only if $\bar{A}$ is connected $(1,k)$-bounded.*

Proof. Let $\bar{A}$ be connected $(1,k)$-bounded for some nonnegative integer k. Theorem 3.1.20 implies that $\bar{A} \models f^k(x) \approx f^k(y)$. Let $a, b \in A$. Then $f^{n+k}(a) = f^n(f^k(a)) = f^n(f^k(b)) = f^k(f^n(b)) = f^k(b)$ which implies that $\bar{A} \models f^{n+k}(x) \approx f^k(y)$. Conversely, suppose $\bar{A} \models f^{n+k}(x) \approx f^k(y)$. Let $a, b \in A$. Then $f^{n+k}(a) = f^k(b)$; so, $a \sim b$ which shows that A is the component in $A/\sim$. Therefore, $\bar{A}$ is connected. Let $a \in A$. Then $b = f(a) \in A$. By the assumption, we have $f(f^k(a)) = f^k(f(a)) = f^k(b) = f^{n+k}(a) = f^k(a)$; so, $\{f^(a)\}$ is a loop in $\bar{A}$. Therefore, $\bar{A}$ is connected $(1,k)$-bounded. □

Corollary 3.1.23. *Let $\bar{A}$ be a mono-unary algebra. Then $\bar{A} \models f^k(x) \approx f^k(y)$ if and only if $\bar{A} \models f^{n+k}(x) \approx f^k(y)$ for all $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$.*

Corollary 3.1.24. *Let $\bar{A}$ be a mono-unary algebra and $k \in \mathbb{N}_0$. If $\bar{A}$ satisfies β_k then $\bar{A}$ satisfies $\gamma_{n,k}$ for all $n \in \mathbb{N}$.*

By Theorem 3.1.20 to Corollary 3.1.24, we can see that all identities of mono-unary algebras are in the form α_k, β_k or $\gamma_{n,k}$.

Let $\sum_1, \sum_2, \sum_3 \sum_\gamma, \sum_\beta, \sum_{\gamma,\beta} \subseteq (W_\tau(X))^2$ be defined as follows:

$$\sum_1 \subseteq \{\alpha_k | k \in \mathbb{N}_0\}$$

$$\sum_2 \subseteq \{\gamma_{n,k} | n \in \mathbb{N}, k \in \mathbb{N}_0\}$$

$$\sum_3 \subseteq \{\beta_k | k \in \mathbb{N}_0\}$$

$$\sum_\gamma = \sum_1 \cup \sum_2$$

$$\sum_\beta = \sum_1 \cup \sum_3$$

$$\sum_{\gamma,\beta} = \sum_1 \cup \sum_2 \cup \sum_3$$

We consider the case that $\sum_1 \neq \phi$ and $\sum_2 \neq \phi$. If $\bar{A}$ is a mono-unary algebra then $\bar{A}$ satisfies identities of the form α_k for all $k \in \mathbb{N}_0$; that is, $\bar{A}$ satisfies $\sum_1$ for all $\sum_1 \neq \phi$. This mean that, if $\sum_1 \neq \phi$ then $Mod(\sum_1)$ is the class of all mono-unary algebras which will be written as $\mathscr{U}$.

Theorem 3.1.25. *Let m be the greatest common divisor of the set $\{n|(\exists k \in \mathbb{N}_0)(\gamma_{n,k} \in \sum_\gamma\}$ and $j = min\{k|(\exists n \in \mathbb{N})(\gamma_{n,k} \in \sum)\}$ then $\bar{A} \in Mod(\sum_\gamma)$ if and only if $\bar{A}$ is an (m, j)-bounded for all $\bar{A} \in \mathscr{U}$.*

Proof. Let $\bar{A}$ be an (m, j)-bounded and let $G = \{(n, k)|(\exists k \in \mathbb{N}_0)(\gamma_{n,k} \in \sum_\gamma\}$. Remark 3.1.15 implies that $\bar{A}$ is (n, k)-bounded for all $(n, k) \in G$. Theorem 3.1.21 implies that $\bar{A}$ satisfies identities of the form $\gamma_{n,k}$ for all $(n, k) \in G$. Since $\bar{A}$ also satisfies $\sum_1$, we have $\bar{A}$ satisfies $\sum_\gamma$; hence, $\bar{A} \in Mod(\sum_\gamma)$. Conversely, let $\bar{A} \in Mod(\sum_\gamma)$. Then $\bar{A}$ satisfies $\sum_\gamma$ which implies that $\bar{A}$ satisfies all identities of the form $\gamma_{n,k}$ for all $(n, k) \in G$. So, Theorem 3.1.21 implies that $\bar{A}$ is (n, k)-bounded for all $(n, k) \in G$. Let $P = \{n|(\exists k \in \mathbb{N}_0)((n, k) \in G)\}$ and $Q = \{k|(\exists n \in \mathbb{N})((n, k) \in G)\}$. Then $\bar{A}$ is (p, q)-bounded for some $m \in \mathbb{N}$ and $q \in \mathbb{N}_0$ such that p is a common divisor of the set P and $q \leq k$ for all $k \in Q$. Since m is the greatest common divisor of P and $j = min(Q)$, we have $p|m$ and $q \leq j$; Theorem 3.1.21 implies that $\bar{A}$ is (m, j)-bounded. □

As a consequence of Theorem 3.1.25, there exist $m \in \mathbb{N}$ and $j \in \mathbb{N}_0$ such that $Mod(\sum_\gamma)$ is the class of all (m, j)-bounded and which will be written as $\mathscr{A}(m, j)$.

Theorem 3.1.26. *Let $p = min\{k|\beta_k \in \sum_3\}$ then $\bar{A} \in Mod(\sum_3)$ if and only if $\bar{A}$ is a connected $(1, p)$-bounded for all $\bar{A} \in \mathscr{U}$.*

Proof. Let $\bar{A}$ be connected $(1, k)$-bounded. Then Remark 3.1.15 suggested that $\bar{A}$ is connected $(1, k)$-bounded for all $k \in \{k|\beta_k \in \sum_3\}$. And by Theorem 3.1.20, $\bar{A}$ satisfies $\sum_3$; hence, $\bar{A}$ satisfies $\sum_\beta$; that is, $\bar{A} \in Mod(\sum_\beta)$. Conversely, if $\bar{A} \in Mod(\sum_\beta)$, then $\bar{A}$ satisfies $\sum_\beta$. Let $H = \{k|\beta_k \in \sum 3\}$. Then $\bar{A}$ satisfies all identities of the form β_k for all $k \in H$. Theorem 3.1.20 implies that $\bar{A}$ is connected $(1, k)$-bounded for all $k \in H$. Therefore, $\bar{A}$ is a connected $(1, p)$-bounded since $p \in H$. □

Theorem 3.1.26 implies that there exists $p \in \mathbb{N}_0$ such that $Mod(\sum_\beta)$ is the class of all connected $(1, p)$-bounded written as $\mathscr{A}_c(1, p)$.

Theorem 3.1.27. *Let $q = min\{k|(\exists n \in \mathbb{N})(\gamma_{n,k} \in \sum_2\} \cup \{k|\beta_k \in \sum_3\}$ then $\bar{A} \in Mod(\sum_{\gamma,\beta})$ if and only if $\bar{A}$ is a connected $(1, q)$-bounded for all $\bar{A} \in \mathscr{U}$.*

Proof. Let $\bar{A}$ be a connected $(1, q)$-bounded for some $q \in \mathbb{N}_0$ and let $H = k|\beta_k \in \sum_3$ and $G = \{(n, k)|(\exists n \in \mathbb{N})(\gamma_{n,k} \in \sum_2)\}$. Remark 3.1.15 implies that $\bar{A}$ is connected $(1, k)$-bounded for all $k \in H$ which shows that $\bar{A}$ satisfies $\sum_3$. By Remark 3.1.15 again, $\bar{A}$ is (n, k)-bounded for all $(n, k) \in G$; so, $\bar{A}$ satisfies $\sum_2$. Therefore, $\bar{A} \in Mod(\sum_{\gamma,\beta})$. Conversely, let $\bar{A} \in Mod(\sum_{\gamma,\beta})$. If $\bar{A}$ is not connected then $\bar{A}$ not satisfies all identities in the form β_k for all $k \in \mathbb{N}_0$; so, $\bar{A} \notin Mod(\sum_{\gamma,\beta})$. Hence, $\bar{A}$ is connected. Let $F = \{k|(\exists n \in \mathbb{N})(\gamma_{n,k} \in \sum_2\} \cup \{k|\beta_k \in \sum_3\}$. By

Remark 3.1.15, $\bar{A}$ is connected $(1,p)$-bounded for some $p \in \mathbb{N}_0$ and $p \leq k$ for all $k \in F$. Since $q = min(F)$, we have $p \leq q$; so, $\bar{A}$ is a connected $(1,q)$-bounded by Remark 3.1.15. □

Theorem 3.1.28. *Let $\mathscr{V}$ be a variety of mono-unary algebras Then,*
(i) $\mathscr{V} = \mathscr{U}$ *or*
(ii) *there exist $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ such that $\mathscr{V} = \mathscr{A}(n,k)$ or*
(iii) *there exist $k \in \mathbb{N}_0$ such that $\mathscr{V} = \mathscr{A}_c(1,k)$.*

Proof. Let $\mathscr{V}$ be a variety of mono-unary algebras. Then there exist a set of identities $\sum$ such that $\mathscr{V} = Mod(\sum)$.
Case 1. If $\sum = \sum_1$ then $\mathscr{V} = \mathscr{U}$.
Case 2. If $\sum = \sum_\gamma$ then there exist $m \in \mathbb{N}$ and $j \in \mathbb{N}_0$ such that $\mathscr{V} = \mathscr{A}(m,,j)$.
Case 3. If $\sum = \sum_\beta$ then there exists $p \in \mathbb{N}_0$ such that $\mathscr{V} = \mathscr{A}_c(1,p)$.
Case 4. If $\sum = \sum_{\gamma,\beta}$ then there exists $q \in \mathbb{N}_0$ such that $\mathscr{V} = \mathscr{A}_c(1,q)$. □

By Theorem 3.1.28, we show all varieties of mono-unary algebras. Let $\bar{\mathscr{L}}$ be a lattice of all varieties of mono-unary algebra. We will explain the lattice $\bar{\mathscr{L}}$ by the theorems below.

Lemma 3.1.29. *Let $m,n \in \mathbb{N}$ and $k,s \in \mathbb{N}_0$ then*

$$\mathscr{A}_c(1,k) \subseteq \mathscr{A}_c(1,s) \text{ if and only if } k \leq s$$

$$\mathscr{A}_c(1,k) \subseteq \mathscr{A}(m,s) \text{ if and only if } k \leq s$$

$$\mathscr{A}_(n,k) \subseteq \mathscr{A}_(m,s) \text{ if and only if } k \leq s \text{ and } n|m.$$

Lemma 3.1.30. *Let $m,n \in \mathbb{N}$ and $k,s \in \mathbb{N}_0$ then*

$$\mathscr{A}_c(1,k) \vee \mathscr{A}_c(1,s) = \mathscr{A}_c(1,max\{k,s\})$$

$$\mathscr{A}_c(1,k) \wedge \mathscr{A}_c(1,s) = \mathscr{A}_c(1,min\{k,s\})$$

$$\mathscr{A}_c(1,k) \vee \mathscr{A}(n,s) = \mathscr{A}(n,max\{k,s\})$$

$$\mathscr{A}_c(1,k) \wedge \mathscr{A}(n,s) = \mathscr{A}(n,min\{k,s\})$$

$$\mathscr{A}(n,k) \vee \mathscr{A}(m,s) = \mathscr{A}([n,m],max\{k,s\})$$

$$\mathscr{A}(n,k) \wedge \mathscr{A}(m,s) = \mathscr{A}((n,m),min\{k,s\})$$

where (a,b) and $[a,b]$ denote the greatest common deviser and the least common factor of a and b respectively.

Proof. Since $k \leq max\{k,s\}$ and $s \leq max\{k,s\}$, Lemma 3.1.29 implies that $\mathscr{A}_c(1,k) \subseteq \mathscr{A}_c(1,max\{k,s\})$ and $\mathscr{A}_c(1,s) \subseteq \mathscr{A}_c(1,max\{k,s\}$ respectively. Then $\mathscr{A}_c(1,max\{k,s\})$ is an upper bound of $\{\mathscr{A}_c(1,k),\mathscr{A}_c(1,s)\}$. Let $\mathscr{A}$ is an upper bound of $\{\mathscr{A}_c(1,k),\mathscr{A}_c(1,s)\}$. Then $\mathscr{A}_c(1,k) \subseteq \mathscr{A}$ and $\mathscr{A}_c(1,k) \subseteq \mathscr{A}$; that is, $\mathscr{A}$ is either $\mathscr{A} = \mathscr{A}_c(1,p)$ for some $p \in \mathbb{N}_0$ where $k \leq p$ and $s \leq p$ or $\mathscr{A} = \mathscr{A}(n,q)$ for some $n \in \mathbb{N}$, $q \in \mathbb{N}_0$ where $k \leq q$ and $s \leq q$. If $\mathscr{A} = \mathscr{A}_c(1,p)$, Lemma 3.1.29 implies that $\mathscr{A}_c(1,max\{k,s\}) \subseteq \mathscr{A}_c(1,p) = \mathscr{A}$ since $max\{k,s\} \leq p$. If $\mathscr{A} = \mathscr{A}(n,q)$, Lemma 3.1.29 implies that $\mathscr{A}_c(1,max\{k,s\}) \subseteq \mathscr{A}(n,q) = \mathscr{A}$ since $max\{k,s\} \leq q$. For any way, we have $\mathscr{A}_c(1,max\{k,s\}) \subseteq \mathscr{A}$; hence, $\mathscr{A}_c(1,max\{k,s\})$ is the least upper bound of $\{\mathscr{A}_c(1,k),\mathscr{A}_c(1,s)\}$. The remaining are prove as the same way. □

Lemma 3.1.29 - 3.1.30 imply that $\bar{\mathscr{L}}$ is a lattice which contains $\mathscr{U}$ and $\mathscr{A}_c(1,0)$ as a greatest element and least element respectively.

Let $\bar{\mathbb{N}}_0^t = (\mathbb{N}_0 \cup \{t\}; R)$ where "R" denote an order defined as follow: aRb if $a \leq b$ for all $a, b \in \mathbb{N}$ and aRt for all $a \in \mathbb{N}_0$.

Let $\bar{\mathbb{N}}_0^t = (\mathbb{N}_0 \cup \{t\}; R)$ where "R" denote an order defined as follow: aRb if $a \leq b$ for all $a, b \in \mathbb{N}$ and aRt for all $a \in \mathbb{N}_0$.

$\bar{\mathbb{D}}_b = (\mathbb{D} \cup \{b\}; R)$ where "R" denote an order defined as follow: aRb if $a|b$ for all $a, b \in \mathbb{D}$ and bRd for all $d \in \mathbb{D}_b$.

Not that $\bar{\mathbb{D}}_b$, $\bar{\mathbb{D}}_0^b$, $\bar{\mathbb{N}}_0^t$, $\bar{\mathbb{D}}_b \times \bar{\mathbb{N}}_0$ and $\bar{\mathbb{D}}_0^b \times \bar{\mathbb{N}}_0^t$ are distributive lattices. Since $(0,t)$ is a greatest element of $\bar{\mathbb{D}}_0^b \times \bar{\mathbb{N}}_0^t$, we obtain that $\bar{\mathbb{D}}_b \times \bar{\mathbb{N}}_0 \cup \{(0,t)\}$ is distributive lattice and $\bar{\mathbb{D}}_b \times \bar{\mathbb{N}}_0 \cup \{(0,t)\}$ is a sublattice of $\bar{\mathbb{D}}_0^b \times \bar{\mathbb{N}}_0^t$.

Theorem 3.1.31. $\bar{\mathscr{L}} \cong ((\bar{\mathbb{D}}_b \times \bar{\mathbb{N}}_0 \cup \{(0,t)\})$.

Proof. Let $\varphi : \bar{\mathscr{L}} \to \bar{\mathbb{D}}_b \times \bar{\mathbb{N}}_0 \cup \{(0,t)\}$ be defined as follows;
$\varphi(\mathscr{U}) = (0,t)$
$\varphi(\mathscr{A}_c(1,k)) = (b,k)$ for all $k \in \mathbb{N}_0$
$\varphi(\mathscr{A}(n,k)) = (n,k)$ for all $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$.

Lemma 3.1.30 implies that φ is an isomorphism and then the theorem is proved. $\square$

Chapter 4

A characterization of all algebras in the variety generated by an irreducible mono-unary algebra

สำนักหอสมุดกลาง

D. Jakubikova-Studenovska [1] has classified all varieties of mono-unary algebras by identities. She observed that there are three forms of varieties of mono-unary algebras which are the class of all mono-unary algebras, classes of (n, k)-bounded and classes of connected $(1, k)$-bounded for all positive integers n and nonnegative integers k. On the other hand, G. Birkhoff proved that a variety is a class of all homomorphic images of subalgebras of direct products of some classes of algebras. In the recent year, Y.Michael observed that there are two forms of subdirectly irreducible mono-unary algebras which are either a loop with non-branched tail or a cycle of prime power length. In this chapter, we prove that a variety of the form connected $(1, k)$-bounded for some nonnegative integer k is generated by a loop with non-branched tail.

4.1 A direct product of a loop with non-branched tail

In this section, we will describe all direct products of a loop with non-branched tail. We prove that a finite product of a loop with non-branched tail $\bar{A}$ is a circle of 2^k-sheets for some nonnegative integer k and a product of $\bar{A}$ is a circle.

Lemma 4.1.1. *Let $(A; f)$ be a loop with non-branched tail . If $|A| = n \geq 2$, there exist $a \in A$ such that $A = \{a, f(a), \ldots, f^{n-1}(a) = f^n(a)\}$ and $f^{-1}(a) = \phi$.*

Proof. Let T be a non-branched tail in (A, f) and let c be a loop in (A, f). Then $t \neq c$ for all $t \in T$ which implies that $f(c) \neq t$ for all $t \in T$; $c \notin f^{-1}(t)$ for all $t \in T$. Hence $|f^{-1}(t)| \leq 1$ for all $t \in T$. Since $A = T \cup \{c\}$ and $|A| = n \geq 2$, we have $|T| = n - 1 \geq 1$; so , $T \neq \phi$. Since $f(x) \in T \cup \{c\}$ for all $x \in T$ and A is connected imply that there exists $b_1 \in T$ such that $f(b_1) = c$. If $f^{-1}(b_1) = \phi$, we have $A = \{b_1, c = f(b_1)\}$; hence $n = 2$ and we are done. By the continuation of

the precess, if $b_k \in f^{-1}(b_{k-1})$ and $f^{-1}(b_k) = \phi$ for a positive integer k then $A = \{b_k, b_{k-1}, \ldots, b_1, c\}$; hence, $n = k+1$ and we are done. But, if $f^{-1}(b_k) \neq \phi$ then the process can be continued. Since $|T| = n-1$, the process must be stopped at the $(n-1)^{th}$ status and $A = \{b_{n-1}, f(b_{n-1}) = b_{n-2}, \ldots, f^{n-1}(b_{n-1}) = \ldots = f(b_1) = c\}$. So, we choose $a = b_{n-1}$. □

Corollary 4.1.2. *Let $(A; f)$ be a loop with non-branched tail and $A = \{a, f(a), \ldots, f^{n-1}(a) = f^n(a)\}$. Then,*
(i) $f^{-1}(f^{n-1}(a)) = \{f^{n-1}(a), f^{n-2}(a)\}$
(ii) $f^{-1}(a) = \phi$
(iii) $f^{-1}(f^{n-1-i}(a)) = \{f^{n-1-(i+1)}(a)\}$ *for all* $1 \leq i \leq n-2$
(iv) $f^t(f^{n-1-i}(a)) = f^{n-1}(a)$ *if and only if* $t \geq i$.

By Lemma 4.1.1, if $(A; f)$ is a loop with non branched tail and $|A| = n$. Then there exists $a \in A$ such that $A = \{a, f(a), \ldots, f^{n-1}(a) = f^n(a)\}$ and $f^{-1}(a) = \phi$. We may assume that $A = \{0, 1, 2, \ldots, n-1\}$ where $f(i) = i-1$ for all $1 \leq i \leq n-1$, $f^{-1}(a) = \phi$ and 0 is a loop of (A, f). Then, by Corollary 4.1.2 we have

(i) $f^{-1}(0) = \{0, 1\}$
(ii) $f^{-1}(n-1) = \phi$
(iii) $f^{-1}(i) = \{i+1\}$ for all $1 \leq i \leq n-2$
(iv) $f^t(i) = 0$ if and only if $t \geq i$
(v) $f^{n-1}(i) = 0$ for all $0 \leq i \leq n-1$.

Definition 4.1.3. Let $(A; f)$ be a mono-unary algebra and $a \in A$ with $f^{-1}(a) \neq \phi$. For each $i \geq 0$, a is said to be e_i-type if $|f^{-1}(a)| = 2^i$. If $|f^{-1}(a)|$ is not finite, we said that a is e_∞-type.

Definition 4.1.4. A circle is a mono-unary algebra $(A; f)$ containing a unique loop of e_∞-type; and for each $a \in A$ with $f^{-1}(f^{-1}(a)) \neq \phi$, the following conditions are satisfied:
(i) if a is e_j-type for some $0 \leq j \leq k$ then $f^{-1}(a)$ contains $\binom{j}{i}$ elements which are e_i-type for all $0 \leq i \leq j$,
(ii) if a is of e_∞-type , $f^{-1}(a)$ contains one element of e_0-type and infinitely many elements of e_∞-type and e_i-type for all $i \in \mathbb{N}$.

For a nonnegative integer k, if a mono-unary algebra $\bar{A}$ contain the unique loop of e_k-type and satisfies the condition (i) of Definition 4.1.4, we call $\bar{A}$ a **circle of 2^k-sheets**.

Definition 4.1.5. Let n and k be nonnegative integers. A circle $(A; f)$ of 2^k-sheets is said to be a **circle of radius n** if
(i) n is the least integer such that $f^n(a)$ is the loop of (A, f) for all $a \in A$,
(ii) for each $a \in A$, if there exists $0 \leq m < n$ such that $f^m(a)$ is the loop of (A, f) then $f^{-1}(a) \neq \phi$.

Example 4.1.6. *A circle and a circle of (2^k)-sheets with radius n for some non-negative integers k and n are shown as the pictures below.*

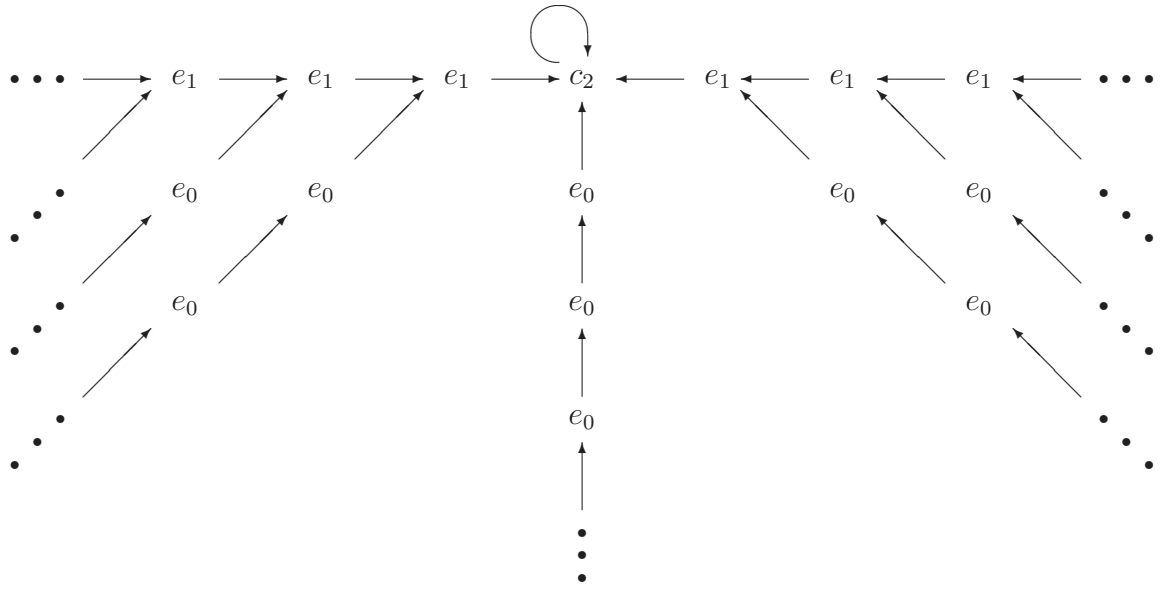

Figure 8. *A circle of* 2^2*-sheets*

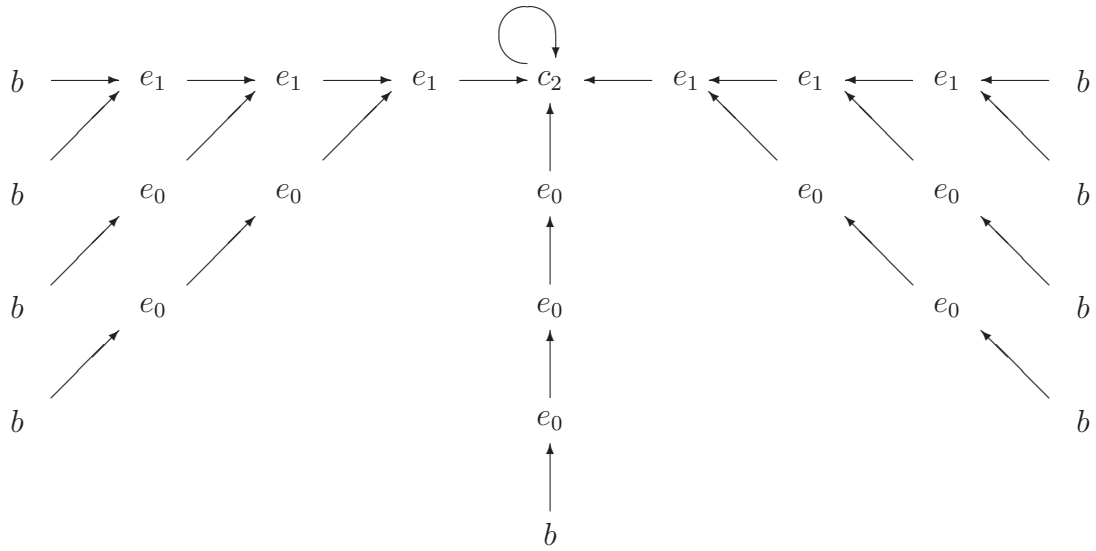

Figure 9. *A circle of* 2^2*-sheets with radius* 4

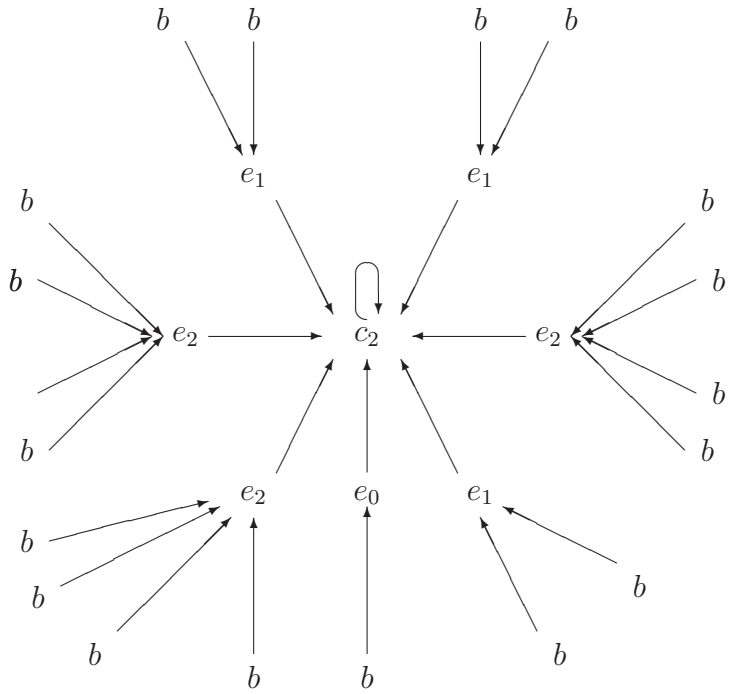

Figure 10. *A circle of* 2^3*-sheets with radius* 2.

Let $(\bar{A}_i)_{i\in I}$ be a family of mono-unary algebras. The direct product $\prod\limits_{i\in I}\bar{A}_i$ of $(\bar{A}_i)_{i\in I}$ is defined as usual. We recall that if $\bar{A}_i = \bar{A}$ for all $i \in I$ we write $\bar{A}^I$ instead of $\prod\limits_{i\in I}\bar{A}_i$; and we denote $(a)_{i\in I}$ by $\bar{a}$. If $J = \phi$ then $\bar{A}^0$ defined to be one-element (trivial) mono-unary algebra. If $I = \{1, 2, \ldots, n\}$, then the direct product can be written as $\bar{A}_1 \times \bar{A}_2 \times \ldots \times \bar{A}_n$. Let $\bar{A}$ be a mono-unary algebra and $P = \bar{A}^I$.

Lemma 4.1.7. *Let* $\bar{A}^I$ *be the product of a loop with non-branched finite tail* $\bar{A}$*. Then*
(i) $\bar{f}(\bar{0}) = \bar{0}$,
(ii) $\bar{f}^{-1}(\bar{a}) = \phi$ *if and only if* $a_j = n-1$ *for some* $j \in I$,
(iii) *for* $\bar{a} \in \bar{A}^I$ *with* $\bar{f}^{-1}(\bar{a}) \neq \phi$*, if there are* $0 \leq m \leq |I|$ *and* $i_1, i_2, \ldots, i_m \in I$ *such that* $a_{i_j} = 0$ *for all* $j = 1, 2, \ldots, m$ *and* $a_i \neq 0$ *otherwise then* $|\bar{f}^{-1}(\bar{a})| = 2^m$,
(iv) *for each* $\bar{a} \in \bar{A}^I$*, if* $\bar{a}$ *is* e_m*-type for some integer* $0 \leq m \leq n-1$ *and* $\bar{f}^{-1}(\bar{f}^{-1}(\bar{a})) \neq \phi$ *then* $\bar{f}^{-1}(\bar{a})$ *contains* $\binom{m}{i}$ *elements which are* e_i*-type for all* $0 \leq i \leq m$.

Proof. Let $|A| = n$ for some positive integer n. (i) It is clear that $\bar{f}(\bar{0}) = \overline{f(0)} = \bar{0}$.
(ii)Assume that $a_j = n-1$ for some $j \in I$. Corollary4.1.2 implies that $f^{-1}(a_j) = \phi$; i.e. $f(a) \neq a_j$ for all $a \in A$; therefore, $\bar{f}^{-1}(\bar{a}) = \phi$. Conversely, if $a_i \neq n-1$ for all $i \in I$ then $a_i < n-1$ for all $i \in I$. Therefore, $f^{-1}(a_i) \neq \phi$ for all $i \in I$. Hence, for each $i \in I$, there exists $b_i \in A$ such that $f(b_i) = a_i$; thus $\bar{f}(\bar{b}) = \bar{a}$ which shows that $\bar{f}^{-1}(\bar{a}) \neq \phi$.
(iii) Let $\bar{a} \in \bar{A}^I$ and $\bar{f}^{-1}(\bar{a}) \neq \phi$. Then $a_i \neq n-1$ for all $i \in I$. Suppose that $a_j = 0$ for some $j \in I$ and $a_i \neq 0$ otherwise. Since $a_i \neq 0$ and $a_i \neq n-1$ for all $i \neq j$, Corollary 4.1.2 implies that $f^{-1}(a_i) = \{a_i + 1\}$. Since $f^{-1}(a_j) = \{0, 1\}$, we have

$\bar{f}^{-1}(\bar{a}) = \{\bar{b} \mid b_i = a_i + 1$ for all $i \neq j$ and $b_j \in \{0,1\}\}$.Therefore, $|\bar{f}^{-1}(\bar{a})| = 2$. If there are $i_1, \ldots, i_m \in I$ for some $0 \leq m \leq |I|$ such that $a_{i_j} = 0$ for all $j = 1, \ldots, m$ and $a_i \neq 0$ otherwise, the same process will imply that $\bar{f}^{-1}(\bar{a}) = \{\bar{b} \mid b_i = a_i + 1$ for all $i \notin \{i_1, \ldots, i_m\}$ and $b_{i_j} \in \{0,1\}$ for all $j = 1, \ldots, m\}$. Therefore, $|\bar{f}^{-1}(\bar{a})| = 2^m$. Moreover, $\bar{f}^{-1}(\bar{a})$ contains $\binom{m}{k}$ tuples whose k components are zero and the others are not zero for all $0 \leq k \leq m$.
(iv) Let $\bar{a} \in \bar{A}^I$ be e_m-type for some nonnegative integer m and $\bar{f}^{-1}(\bar{f}^{-1}(\bar{a})) \neq \phi$. Then $|\bar{f}^{-1}(\bar{a})| = 2^m$. If there exists $j \in I$ such that $a_j = n-2$ then $f^{-1}(a_j) = \{n-1\}$; so, for each $\bar{b} \in \bar{f}^{-1}(\bar{a})$ there is $j \in I$ such that $b_j = n-1$. Thus, $\bar{f}^{-1}(\bar{b}) = \phi$ for all $\bar{b} \in \bar{f}^{-1}(\bar{a})$; i.e. $\bar{f}^{-1}(\bar{f}^{-1}(\bar{a})) = \phi$ which is a contradiction. Therefore, $a_i \neq n-2$ for all $i \in I$. Hence, $\bar{f}^{-1}(\bar{b}) \neq \phi$ for all $\bar{b} \in \bar{f}^{-1}(\bar{a})$. By (iii), $\bar{f}^{-1}(\bar{a})$ contains $\binom{m}{i}$ elements which are e_i-type for all $0 \leq i \leq m$.

□

Theorem 4.1.8. *Let $\bar{A} = (A, f)$ be a loop with non-branched finite tail. The finite product A^I is the circle of $2^{|I|}$-sheets with radius n-1.*

Proof. Let $|A| = n$ for some positive integer n. Lemma 4.1.7 (i) implies that $\bar{0}$ is the loop of $\bar{A}^I$. Since 0 is the loop in $\bar{A}$, we have that $a_i = 0$ for all $i \in I$ if $\bar{f}(\bar{a}) = \bar{a}$; so, $\bar{a} = \bar{0}$ which shows that $\bar{0}$ is the unique loop in $\bar{A}^I$. Let $\bar{a} \in \bar{A}^I$ be e_m-type for some nonnegative integer m and $\bar{f}^{-1}(\bar{f}^{-1}(\bar{a})) \neq \phi$. By Lemma 4.1.7 (iv), the condition (ii) of Definition 4.1.4 holds. Therefore, $\bar{A}^I$ is the circle of $2^{|I|}$-sheets. Let $\bar{a} \in \bar{A}^I$. Then Corollary 4.1.2 (v) shows that $f^{n-1}(a_i) = 0$ for all $i \in I$; therefore, $\bar{f}^{n-1}(\bar{a}) = (f^{n-1}(a_i))_{i \in I} = (0)_{i \in I} = \bar{0}$; but, $\bar{f}^{n-2}((n-1)_{i \in I}) = (f^{n-2}(n-1))_{i \in I} = (n-1-(n-2)_{i \in I} = (1)_{i \in I} \neq \bar{0}$ shows that n-1 is the least integer such that $\bar{f}^{n-1}(\bar{a}) = \bar{0}$ for all $\bar{a} \in \bar{A}^I$. If $\bar{f}^m(\bar{a})$ is a loop for some integer $0 \leq m \leq n-1$ then $f^m(a_i) = 0$ for all $i \in I$, Corollary 4.1.2 (iv) implies that $a_i < m < n-1$ for all $i \in I$; so, $\bar{f}^{-1}(\bar{a}) \neq \phi$ by Lemma 4.1.7 (ii). Hence, $\bar{A}^I$ is the circle of $2^{|I|}$-sheets with radius n-1. □

If $\bar{A}$ is a loop with non-branched tail whose $|A| = n$, the same argument in the proof of Theorem 4.1.8 shows that the infinite product $\bar{A}^I$ of $\bar{A}$ is a circle with radius n-1.

4.2 A subalgebra of a circle

Recall that a mono-unary algebra $\bar{B} = (B, g)$ is a subalgebra of (A,f) if $B \subseteq A$ and $f|_B = g$; or equivalently, if $f(b) \in B$ for all $b \in B$.

Theorem 4.2.1. *Every subalgebra of a circle contains the loop of the circle.*

Proof. Let $\bar{B}$ be a subalgebra of $\bar{A}$. The connectedness of $\bar{A}$ implies that there exists a nonnegative integer m such that $f^m(b)$ is the loop of $\bar{A}$ for all $b \in B$. Since $B \neq \phi$, the loop of $\bar{A}$ is in B. □

Note that: a subalgebra of a circle is not necessary a circle. We will describe all subalgebras of a circle as follows.

Definition 4.2.2. Let $\bar{A} = (A; f)$ be a circle with the loop c. For each $a \in A$, the set $\{b \in A | \exists (m \in \mathbb{N}_0)(f^m(b) = a)\}$ is called the **down set** of a; and we denote it by $\downarrow a$.

For each $a \in f^{-1}(c) \setminus \{c\}$, the set $\downarrow a \cup \{c\}$ is called a **sheet** of $\bar{A}$. A **subsheet** S of a sheet W is a subset of W which is closed under f. It is easy to prove the following lemma.

Lemma 4.2.3. *Every sheet and subsheet of a circle are subalgebras of the circle.*

Proof. Let S be a sheet of $\bar{A}$ and c be a loop of $\bar{A}$. Then $S = \downarrow a \cup \{c\}$ for some $a \in f^{-1}(c) \setminus \{c\}$. Let $s \in S$ with $f(s) \neq c$. Then $s \neq c$; so, $s \in \downarrow a$; there exist a nonnegative integer m such that $f^m(s) = a$. Since $f^{m-1}(f(s)) = f^m(s) = a$, we have $f(s) \in \downarrow a$. Therefore, $f(s) \in S$. □

Definition 4.2.4. Let $\bar{B}$ and $\bar{C}$ be subalgebras of a mono-unary algebra $\bar{A}$. We denote by $\bar{B} \cup \bar{C}$; the **union algebra** of $\bar{A}$ whose universe is $B \cup C$ and the unary operation f on $B \cup C$ is defined as follow; $f(x) = f^{\bar{B}}(x)$ if $x \in B$ and $f(x) = f^{\bar{C}}(x)$ if $x \in C \setminus B$. We note that, if $\bar{B}$ and $\bar{C}$ are pairwise disjoint then an operation f on $B \cup C$ is $f|_{\overline{B \cup C}}$.

From the Deffinition 4.2.4, one can notice that a union algebra of a circle is a union algebra of subsheets of the circle.

Example 4.2.5. *We will show examples of a sheet and a subseet and a subalgebra of a circle in the pictures below.*

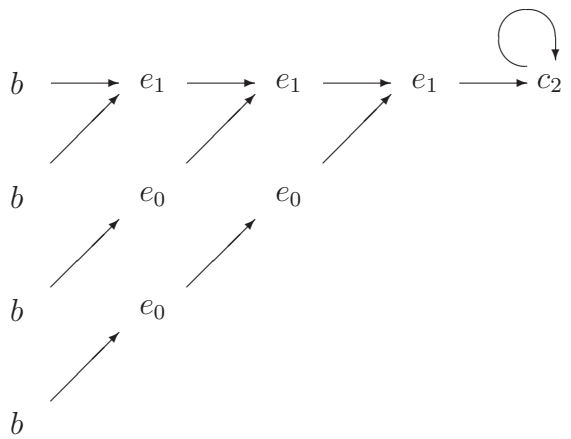

Figure 11. *A sheet of a circle of* 2^2*-sheets with radius* 4

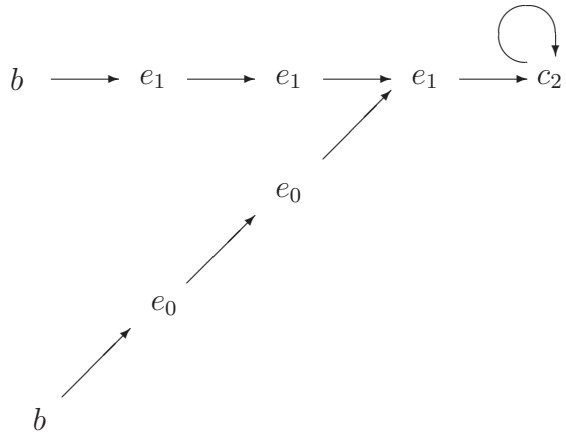

Figure 12. *A subsheet of a circle of* 2^2*-sheets with radius* 4

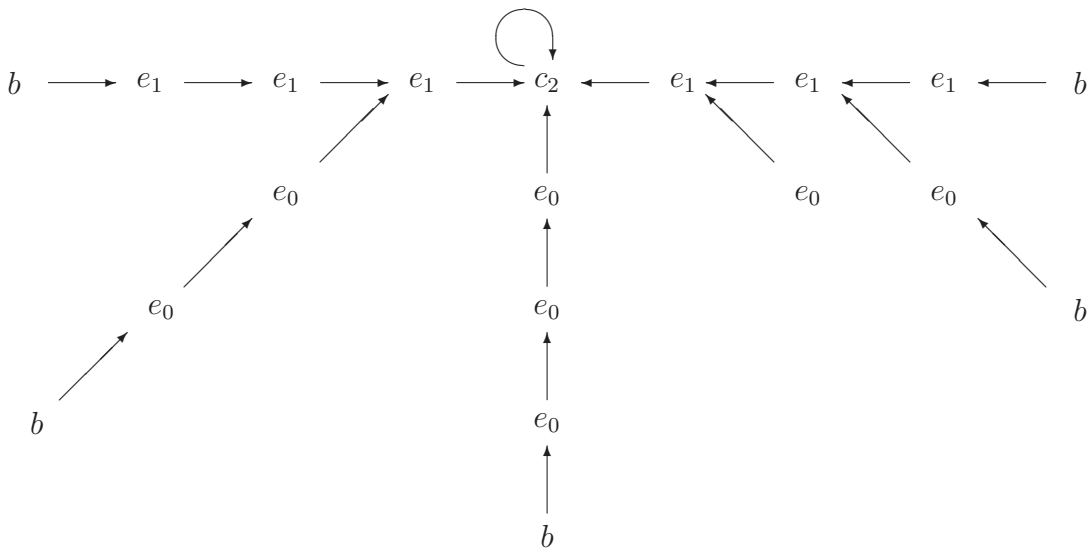

Figure 13. *A subalgebra of a circle of* 2^2*-sheets with radius* 4

Lemma 4.2.6. *For a nonnegative integer k, every subalgebra of connected (1,k)-bounded is connected (1,k)-bounded.*

Proof. Let $\bar{A}$ be a connected (1,k)-bounded with the loop c and let $\bar{B}$ be a subalgebra of $\bar{A}$. The connectedness of $\bar{A}$ implies that $f^k(b) = c$ for all $b \in B$. Since $B \neq \phi$, we have $c \in B$. Therefore, $\bar{B}$ is a connected (1,k)-bounded. □

4.3 A homomorphic image of a circle

Theorem 4.3.1. *Let* $\bar{A} = (A, f)$ *be a loop with non-branched finite tail.Then* A^I *is a connected* $(1, |A| - 1)$*-bounded.*

Proof. Let $\bar{a}, \bar{b} \in \bar{A}^I$. Then $\bar{f}^{n-1}(\bar{a}) = \overline{f^{n-1}(a)} = \bar{0} = \overline{f^{n-1}(b)} = \bar{b}$. Therefore, $\bar{A}^I \models \bar{f}^{n-1}(\bar{a}) \approx \bar{f}^{n-1}(\bar{b})$ which implies that $\bar{A}^I$ is a connected (1,n-1)-bounded. □

Let (A, f) and (B, g) be mono-unary algebras. Recall that a function $h : A \to B$ is called a homomorphism of A into B if $h(f(a)) = g(h(a))$ for all $a \in A$. If h is surjective then the homomorphism h is called an epimorphism from A onto B; and h is call an isomorphism from A to B if h is bijective.

Proposition 4.3.2. *Let $\bar{A} = (A; f)$ and $\bar{B} = (B, g)$ be mono-unary algebras and $h : \bar{A} \to \bar{B}$ be a homomorphism. If $\bar{A}$ is connected $(1, k)$-bounded for some nonnegative integer k then*
(i) *$h(c)$ is a loop in $\bar{B}$ if c is loop of $\bar{A}$,*
(ii) *if $f(a) = b$ for some $a, b \in A$ then $g(h(a)) = h(b)$.*

Proof. (i) Since $g(h(c)) = h(f(c)) = h(c)$; hence, $h(c)$ is the loop in $\bar{B}$.
(ii) Let $a, b \in A$ with $f(a) = b$. Then $h(b) = h(f(a)) = g(h(a))$. □

Theorem 4.3.3. *A homomorphic image of connected $(1, k)$-bounded is connected $(1, k)$-bounded for all nonnegative integer k.*

Proof. Let $\bar{B}$ be a homomorphic image of $\bar{A}$ by a homomorphism h and let $c, d \in B$. Then there exist $a, b \in A$ such that $f(a) = c$ and $f(b) = d$. Since $\bar{A} \models f^k(x) \approx f^k(y)$, we have $g^k(x) = g^k(h(a)) = h(f^k(a)) = h(f^k(b)) = g^k(h(b)) = g^k(y)$ which implies that $\bar{B} \models f^k(\ x) \approx f^k(y)$; so, $\bar{B}$ is connected $(1, k)$-bounded. □

4.4 A Variety generated by a loop with non-branched tail

Let $\bar{A}$ be an algebra. It is well known that the variety generated by $\bar{A}$ is the class of all algebras which are homomorphic images of subalgebras of products of $\bar{A}$. As we discuss in section 1.1, a product of a loop with non-branched tail is a circle or a circle of 2^n-sheets for some nonnegative integer n. If we would like to describe all algebras in the variety generated by a loop with non-branched tail, it is left to describe all homomorphic images of a circle.

Let $\bar{A}$ be a loop with non-branched finite tail. Then $\bar{A}^I$ is a circle with radius $|A| - 1$ for an index set I. Let $\bar{B}$ be a $(1, |A| - 1)$-bounded with the loop 0 and let $h : \bar{B} \to \bar{A}^I$ with $h(0) = \bar{0}$. If $|A| = 1$ or $|B| = 1$, it is clear that h is a homomorphism. We consider the case $|A| > 1$ and $|B| > 1$. If I is infinite, A^I contains infinitely many elements of e_∞-type. For each integer $1 \leq m \leq |A| - 2$ and $a \in (f^{-1})^m(0)$, the number of elements of e_∞-type is large enough to choose $h(a) \in (\bar{f}^{-1})^m(\bar{0})$ such that $h(a)$ is e_∞-type of A^I and $h(a) \neq h(b)$ for all $b \in (f^{-1})^m(0) - \{a\}$. Since $\bar{B}$ is connected and $h(a)$ is e_∞-type for all $a \in \bar{B} \setminus (f^{-1})^{|A|-1}(0)$, we can choose $h(a)$ such that $\bar{f}(h(a)) = h(f(a))$ for all $a \in B$. By this way, one can see that h is a monomorphism. Therefore, $\bar{B}$ can be embedded as a subalgebra of $\bar{A}^I$. We conclude in the following theorem.

Theorem 4.4.1. *The variety generated by a loop $\bar{A}$ with non-branched finite tail contains precisely all mono-unary algebras which is connected $(1, |A|-1)$-bounded.*

4.5 A Variety generated by a cycle

In this section, we will describe all algebras in a variety generated by a cycle with prime length which are those homomorphic images of subalgebras of products of a cycle with prime length.

Theorem 4.5.1. *The following statements are hold.*

(i) *A product of a cycle with finite length n is $(n, 0)$-bounded.*

(ii) *A homomorphic image of a (n,0)-bounded is (n,0)-bounded for all positive integer n.*

Proof. (i) Let $\bar{A}$ be a cycle with length n for some positive integer n and let $\bar{a} \in A^I$. Since $\bar{f}^n(\bar{a}) = \bar{f}^n((a_i)_{i\in I}) = (f^n(a_i))_{i\in I} = (a_i)_{i\in I} = \bar{a}$, we have $\bar{A}^I \models f^n(x) \approx x$. Therefore, $\bar{A}^I$ is $(n, 0)$-bounded.
(ii) Let $\bar{A}$ be a $(n, 0)$-bounded for some positive integer n and $h : \bar{A} \to \bar{B}$ be an epimorphism. Since $\bar{A} \models f^n(x) \approx x$, we have $g^n(h(a)) = h(f^n(a)) = h(a)$ for all $a \in A$ which shows that $\bar{B} \models f^n(x) \approx x$; so, $h(\bar{A})$ is (n,0)-bounded. □

We note that every element in a product $\bar{A}^I$ of a cycle $\bar{A}$ with length n is a cyclic in a cycle of length n in $\bar{A}^I$. Therefore, every component in $\bar{A}^I$ is a cycle of length n. $|A^I| = n^{|I|}$ implies that $\bar{A}^I$ contains $n^{|I|-1}$ cycles of length n in $\bar{A}^I$. Hence, we can conclude that, an infinite product of a cycle of length n contains infinitely many cycles of length n for all positive integer n.

Example 4.5.2. *A finite product of a cicile with finite length are shown as follows:*

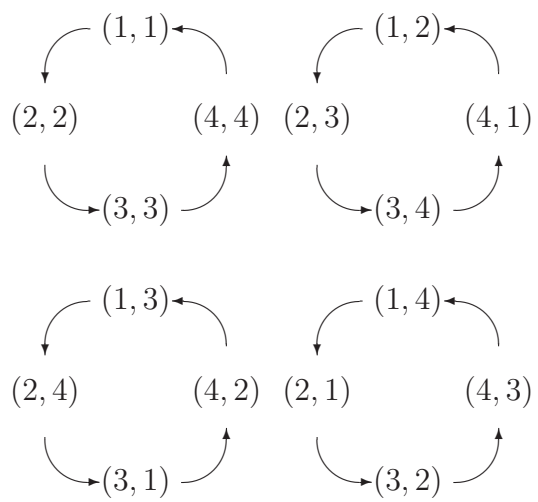

Figure 14.*A product $\bar{A}^2$ of a cycle $\bar{A}$ with length* 4

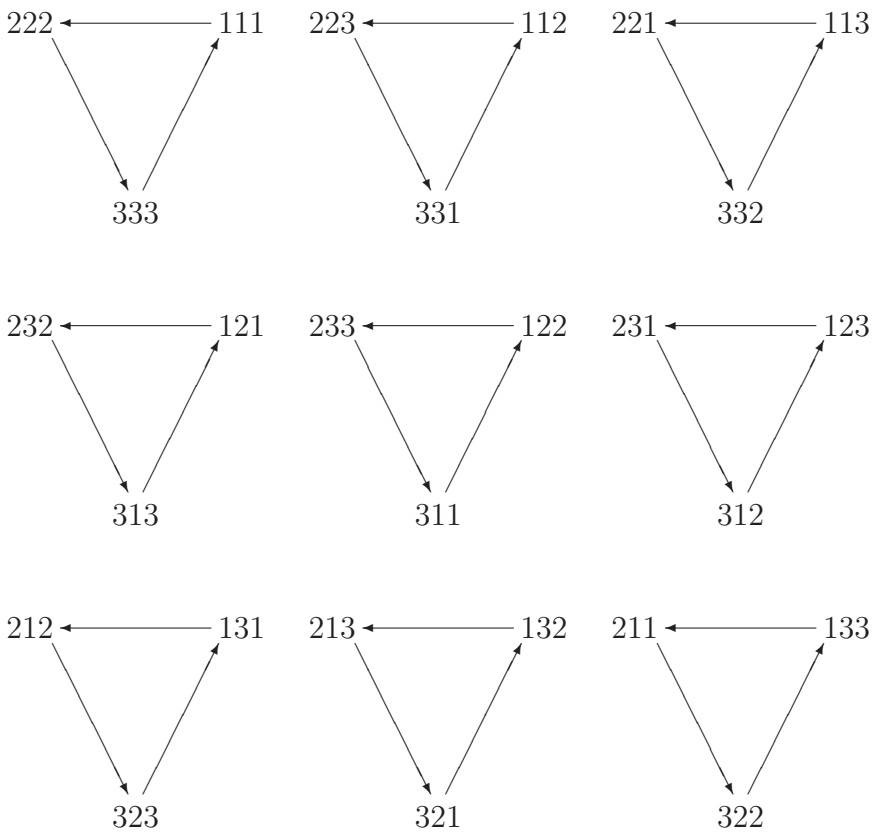

Figure 15. *A product* $\bar{A}^3$ *of a cycle* $\bar{A}$ *with length* 3

Since every cycle in a mono-unary algebra $\bar{A} = (A, f)$ is closed under f; so, every cycle is a subalgebra of a mono-unary algebra. However, every proper subset of a cycle in $\bar{A}$ is not a subalgebra of $\bar{A}$; We have the following lemma.

Lemma 4.5.3. *A cycle in a product* $\bar{A}^I$ *of a cycle with finite length* $\bar{A}$ *is a subalgebra of* $\bar{A}^I$.

Lemma 4.5.4. *Let* $\bar{B}$ *be subalgebra of a product of a cycle with length* n *for some positive integer* n. *Then,*
(i) $\bar{B}$ *is a union of cycles in* $\bar{A}^I$,
(ii) $\bar{B}$ *is* $(n, 0)$*-bounded.*

Lemma 4.5.5. *For a positive integer* n, *a connected* $(n, 0)$*-bounded is a homomorphic image of a cycle with length* n.

Proof. Let $\bar{A} = (A, f)$ be a connected $(n, 0)$-bounded and $\bar{B} = (B, g)$ be a cycle with length n for some a positive integer n. Then there exists a positive integer m such that $m|n$ and $\bar{A}$ is a cycle with length m. So, there are $a \in A$ and $b \in B$ such that $A = \{a, f(a), \ldots, f^{m-1}(a)\}$ and $B = \{b, g(b), \ldots, g^{n-1}(b)\}$. Let $h : B \to A$ be defined as follows: $h(b) = a$ and $h(g^i(b)) = f^i(a)$ for all $0 \leq i \leq n-1$. Then h is well-defined. If $c \in B$, there exists a nonnegative integer v such that $c = g^v(b)$; so, $h(g(c)) = h(g^{v+1}(b)) = f^{v+1}(a) = f(f^v(a)) = f(h(g^v(b))) = f(h(c))$ which shows that h is a homomorphism. Since $m \leq n$, h is surjective. $\square$

Let $\bar{A}$ be a cycle with finite length n for some positive integer n. Then for an index set I, $\bar{A}^I$ is $(n,0)$-bounded whose all components are cycles with length n. Let $\bar{B}$ be a $(n,0)$-bounded algebra. Then every component in $\bar{B}$ is connected $(n,0)$-bounded. If $\bar{B}$ has only one component, Lemma 4.5.5 implies that $\bar{B}$ is a homomorphic image of a subalgebra of $\bar{A}^I$. We consider the case that $\bar{B}$ has more than one components. If I is infinite, $\bar{A}^I$ contains infinitely many components which are a cycle of $\bar{A}^I$ with length n. Lemma 4.5.5 implies that for each component C in $\bar{B}$, there is an epimorphism h_C from some component in $\bar{A}^I$ to C. Since all components are pairwise disjoin, the function $h = \bigcup_{C \subseteq \bar{B}} h_C$ is a homomorphism from a subalgebra of $\bar{A}^I$ to $\bar{B}$. Therefore, $\bar{B}$ is a homomorphic image of a subalgebra of $\bar{A}^I$. We conclude in the following theorem.

Theorem 4.5.6. *The variety generated by a cycle $\bar{A}$ with finite length contains precisely all $(|A|,0)$-bounded mono-unary algebras.*

Bibliography

[1] D. Jakubikova-Studenovska (2009). *Mono-unary Algebras,* Safarik University in Kosice, Faculty of Science, Institute of Mathematics.

[2] G.Birkhoff (1940, 1948, 1967). *Lattice Theory,* American Mathematical Society, Colloqium Publications, Vol 25, 63.

[3] Y.Michael (October 1967). *Subdirectly Irreducible Unary Algebras,* The American Mathematical Monthly,Vol 74, No 8, 957-960.

[4] B.A.Davey and H.A.Priestley (1990). *Introduction to Lattices and Order*, Cambridge Mathematical Textbooks, New York.

[5] K. Denecke and S.L. Wismath (2002). *Universal Algebra and Application,* Chapman & Hall/CRC.

Biography

NAME	Mr. Sujin Sukgumpapan
ADDRESS	91 Moo 9 Tumbon Nongnguluam A. Mueang, Nakhonpathom, 73000

INSTITUTION ATTENDED

2001	Bachelor of Science Program in Petrochemical and Polymer, Silpakorn University
2010	Master of Science in Apply Statistic, Silpakorn University

สำนักหอสมุดกลาง
มหาวิทยาลัยศิลปากร