DUALITY OF SEQUENCE SPACES OF INFINITE MATRICES

By

Mr. Suchat Samphavat

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree
Master of Science Program in Mathematics
Department of Mathematics
Graduate School, Silpakorn University
Academic Year 2011

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The Graduate School, Silpakorn University has approved and accredited the Thesis title of “Duality of sequence spaces of infinite matrices” submitted by MR. Suchat Samphavat as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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In this thesis, we define, for each $1 \leq r < \infty$, the set $\mathcal{F}'$ of infinite complex matrices as follows:

$$\mathcal{F}' := \left\{ \left[ a_{jk}^{(k)} \right]_{j,k=1}^{\infty} : \left\| \sum_{k=1}^{n} a_{jk}^{(k)} e^j \right\| \leq B \left( l^2 \right) \right\}.$$ 

We first show as a preliminary that equipped with the norm

$$\left\| \left[ a_{jk}^{(k)} \right]_{j,k=1}^{\infty} \right\| := \left\| \sum_{k=1}^{n} a_{jk}^{(k)} e^j \right\|^{1/r},$$

the set $\mathcal{F}'$ is a Banach space. The main goal of this research is to decompose the dual $\left( \mathcal{F}' \right)^*$ of $\mathcal{F}'$ as an $l^1$ direct-sum of its two closed subspaces by a way analogous to the classical theorem of Dixmier on decomposing the dual $B \left( l^2 \right)^*$ of $B \left( l^2 \right)$. 
ในวิทยานิพนธ์นี้ เรามีสมมติฐาน $3'$ ของเอนโทรปีนั้นค่อนข้างกว้างขึ้น สำหรับ $1 \leq r < \infty$ ดังนี้

$$3' := \left\{ \left( \left[ a^{(k)}_{\mu} \right] \right)_{k=1}^{\infty} : \left\| \sum_{k=1}^{\infty} a^{(k)}_{\mu} \right\| \in B(I^2) \right\}$$

ในขั้นแรก เราได้แสดงว่าสมมติฐาน $3'$ พรมตัวแปรที่นิยมโดย

$$\left\| \left( \left[ a^{(k)}_{\mu} \right] \right)_{k=1}^{\infty} \right\| := \left( \sum_{k=1}^{\infty} \left| a^{(k)}_{\mu} \right|^r \right)^{1/r}$$

เป็นปริบรมบานาค จุดประสงค์หลักของวิทยานิพนธ์นี้คือการแยกปริบรมบานาค $(3')'$ ของ $3'$ เป็น ส่วน ในรูปของ $I'$ พรมตัวแปรของ $2$ ปริบรมบานาคของ $(3')'$ ในทั้งหมดแต่ยังกับทฤษฎีบท ของคิชเชมิเอร์ที่ว่าตัวการแยกปริบรมบานาค $B(I^2)$ ของ $B(I^2)$
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Chapter 1

Introduction and Preliminaries

A beautiful decomposition of the dual $\mathcal{B}(l^2)^*$ of the Banach algebra $\mathcal{B}(l^2)$ of bounded linear operators on $l^2$, under the usual multiplication, was established by Dixmier (see [5] and [6]). He proved that every bounded linear functional $f$ in $\mathcal{B}(l^2)^*$ can uniquely be decomposed as the sum $f = g + h$ of two bounded linear functionals $g$ and $h$ on $\mathcal{B}(l^2)$ such that $g$ is a trace functional defined associated with a trace class operator $T$ by $g(S) = \text{trace}(ST)$ for all $S \in \mathcal{B}(l^2)$, and $h$ vanishes on the ideal $\mathcal{K}(l^2)$ of compact operators on $l^2$. The most interesting part of the theorem of Dixmier mentioned above is that the norm of the decomposition $f = g + h$ of each $f$ in $\mathcal{B}(l^2)^*$ is additive, i.e., $\|f\| = \|g\| + \|h\|$. An part of Schatten’s theorem (see [12]) states that $\mathcal{K}(l^2)^* \cong \mathcal{C}^1$, where $\mathcal{C}^1$ denotes the class of all trace class operators on $l^2$. From the theorem of Schatten, it can easily be deduced that the space of all trace functionals on $\mathcal{B}(l^2)$ is isometrically isomorphic to the dual $\mathcal{K}(l^2)^*$ of $\mathcal{K}(l^2)$. Thus the theorem of Dixmier mentioned above can be symbolized as $\mathcal{B}(l^2)^* = \mathcal{K}(l^2)^* \oplus \mathcal{K}(l^2)_s$, where $\mathcal{K}(l^2)_s$ denotes the space of all linear functionals on $\mathcal{B}(l^2)$ vanishing on $\mathcal{K}(l^2)$, which are called singular functionals on $\mathcal{K}(l^2)$, and the notation “$\oplus$” is referred to as the $l^1$ direct-sum.

Let $1 \leq p, q, r < \infty$. An infinite scalar matrix $A = [a_{jk}]$ is said to define a linear operator from $l^p$ into $l^q$ if for every $x = \{x_k\}_{k=1}^\infty$ in $l^p$ the series $\sum_{k=1}^\infty a_{jk}x_k$ converges for all $j$, and the sequence $Ax := \left\{\sum_{k=1}^\infty a_{jk}x_k\right\}_{j=1}^\infty$ is a member of $l^r$. If a matrix $A$ defines a linear operator from $l^p$ into $l^q$, we then call the operator $x \mapsto Ax$ the linear operator defined by $A$. In this case, it can be shown by the uniform boundedness principle that the linear operator defined by $A$ is bounded. Let $\mathcal{M}(l^p, l^q)$ be the set of all infinite matrices which define linear operators from $l^p$ into $l^q$. For each matrix $A$, we call $A$ a bounded matrix and define $\|A\|$ to be the norm of the linear operator defined by $A$ if $A \in \mathcal{M}(l^p, l^q)$ and call $A$ an unbounded matrix and define $\|A\|$ to be $\infty$ otherwise. It is well-known that $\mathcal{M}(l^p, l^q)$ is a Banach space under the norm $\|\cdot\|$. Indeed, it coincides with the set of matrix representations of all bounded linear operators from $l^p$ into $l^q$ with respect to the standard Schauder bases of $l^p$ and $l^q$, which is isometrically isomorphic to the Banach space $\mathcal{B}(l^p, l^q)$ of all bounded linear operators from $l^p$ into $l^q$. A matrix $A$ is called a compact matrix if the linear operator defined by $A$ is a compact operator.
For each matrix $A = [a_{ji}]$ and positive integer $n$, let $A_n = [b_{ji}]$ be the matrix with $b_{ji} = a_{ji}$ for all $1 \leq j, i \leq n$ and $b_{ji} = 0$ otherwise, and let $A_n = [c_{ji}]$ be the matrix with $c_{ji} = a_{ji}$ for all $j, i \geq n$ and $c_{ji} = 0$ otherwise. The following are well-known facts about infinite matrices which are useful for the research.

**Theorem 1.1.**

1. If $[a_{ji}]$ and $[b_{ji}]$ are scalar matrices such that $|a_{ji}| \leq b_{ji}$ for all $j, i$, then $\| [a_{ji}] \| \leq \| [b_{ji}] \|$.

2. A matrix $A$ belongs to $\mathcal{B}(l^p, l^q)$ if and only if $\sup_n \| A_n \| < \infty$.

3. For every matrix $A$, $\| A_n \| \not\to \| A \|$.

4. For each $A \in \mathcal{B}(l^p)$ and positive integer $n$, $\| A_n + A_n \| = \max\{\| A_n \|, \| A_n \|\}$.

5. A matrix $A$ is compact as an operator on $l^2$ if and only if $\| A_n - A \| \to 0$.

The Schur product or Hadamard product or entry-wise product of two scalar matrices $A = [a_{jk}]$ and $B = [b_{jk}]$ having the same size is defined by the matrix $A \odot B := [a_{jk}b_{jk}]$. In [13], Schur proved that Banach space $\mathcal{B}(l^2)$ is a commutative Banach algebra (without identity) under the operator norm and the Schur product multiplication. After that, Bennett extended in [1] the result of Schur referred to above. He showed for each $1 \leq p, q < \infty$ that the Banach space $\mathcal{B}(l^p, l^q)$ under the Schur product operation is also a Banach algebra. These beautiful results of Bennett motivated Chaisuriya and Ong [2] to study some classes of infinite matrices over Banach algebras with identity. In [2], for a fixed Banach algebra $\mathcal{B}$ with identity and $1 \leq p, q, r < \infty$, the authors defined the class $S^{p,q,r}_{\mathcal{B}}$ of matrices $A = [a_{jk}]$ over $\mathcal{B}$ such that the absolute Schur $r$-th power $A^r := \| a_{jk} \|^r$ defines a linear operator from $l^p$ into $l^q$. And then they proved that it is a Banach algebra under the absolute Schur $r$-norm defined by

$$\| A \|_{p,q,r} = \| A^r \|^{1/r}$$

and the Schur product, which is straightforwardly generalized to the setting of matrices over the Banach algebra $\mathcal{B}$ by using the multiplication in $\mathcal{B}$. The authors also provided a beautiful relationship, which follows from the results of Schur and Bennett mentioned above, between the algebra $\mathcal{B}(l^p, l^q)$ of all bounded operators form $l^p$ into $l^q$ and the algebra $S^{p,q}_{\mathcal{B}}(C)$. They found that $\mathcal{B}(l^p, l^q)$ is contained in $S^{p,q}_{\mathcal{B}}(C)$ as a non-closed ideal for all $r \geq 2$.

In [8], Livshits, Ong and Wang studied the duality of the absolute Schur algebras $S^{2,2}_{\mathcal{B}}(C)$ by a way analogous to Dixmier’s theorem and Schatten’s theorem mentioned in the first paragraph. The authors defined the class $\mathcal{K}^r$ of infinite matrices $A$ such that $A^r$ is compact as an operator on $l^2$ for playing the role as the class $\mathcal{K}(l^2)$ of all compact operators on $l^2$. They also constructed a class $\mathcal{M}^r$ of infinite matrices for playing the role as the class $C^1$ of all trace class operators, which is known as the dual of $\mathcal{K}(l^2)$. They obtained that $(\mathcal{K}^r)^* \cong \mathcal{M}^r$ and that each bounded
linear functional \( \varphi \) on \( S^r_{2,2}(C) \) can uniquely be decomposed as the sum \( \varphi = \rho + \psi \), where \( \rho \) is determined by a unique matrix in \( \mathcal{M}^r \) under a certain way and \( \psi \) is a singular functional on \( \mathcal{K}^r \). Furthermore, the decomposition \( \varphi = \rho + \psi \) satisfies \( \| \varphi \| = \| \rho \| + \| \psi \| \). Schatten's theorem also states that the trace class operators form a predual of \( \mathcal{B}(l^2) \). An analogue of this result on the setting of Livshits, Ong and Wang: \( (\mathcal{M}^r)^* \cong S^r_{2,2}(C) \), was also obtained.

From the beautiful result of Chaisuriya and Ong that the absolute Schur algebra \( S^r_{2,2}(C) \) contains \( B(l^2) \) as a non-closed ideal, Rakbud and Ong defined three sequence spaces of matrices from \( S^r_{2,2}(C) \) in [11] as follows:

\[
\Omega_b = \left\{ \{A_k\}_{k=1}^{\infty} \subseteq S^r_{2,2}(C) : \text{the sequence } \left\{ \sum_{k=1}^{n} \left| A_{k}^{[2]} \right| \right\}_{n=1}^{\infty} \text{ is bounded in } B(l^2) \right\},
\]

\[
\Omega_c = \left\{ \{A_k\}_{k=1}^{\infty} \subseteq S^r_{2,2}(C) : \text{the sequence } \left\{ \sum_{k=1}^{n} \left| A_{k}^{[2]} \right| \right\}_{n=1}^{\infty} \text{ converges in } B(l^2) \right\},
\]

and

\[
\Omega_\kappa = \left\{ \left\{ a_{ji}^{(k)} \right\}_{k=1}^{\infty} \subseteq S^r_{2,2}(C) : \text{the matrix } \left[ \sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^2 \right] \in \mathcal{K}(l^2) \right\}.
\]

The authors obtained the inclusion relation among these three spaces as follows: \( \Omega_\kappa \subsetneq \Omega_c \subsetneq \Omega_b \). They defined naturally a norm on these three spaces by

\[
\| \{A_k\}_{k=1}^{\infty} \| = \left( \sup_n \left( \sum_{k=1}^{n} \left| A_{k}^{[2]} \right| \right)^2 \right)^{1/2}
\]

and showed that all three sequence spaces equipped with this norm are Banach spaces. It was observed that because of the non-closedness of \( B(l^2) \) in \( S^r_{2,2}(C) \), the restrictions of these sequence spaces to \( B(l^2) \) are all not complete. The study on this paper was mainly focused on the sequence spaces \( \Omega_c \) and \( \Omega_\kappa \). The authors studied sequential convergence in these two sequence spaces and duality and preduality of \( \Omega_\kappa \).

From the idea of Rakbud and Ong referred to above, we obtain a way analogous to the classical sequence spaces \( l^p \) to define sequence spaces of infinite matrices as follows. Let \( \mathcal{M}_\infty \) be the vector space of all infinite complex matrices. For each \( 1 \leq r < \infty \), let

\[
\mathcal{L}^r = \left\{ \left\{ a_{ji}^{(k)} \right\}_{k=1}^{\infty} \subseteq \mathcal{M}_\infty : \left[ \sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^r \right] \in B(l^2) \right\}.
\]

In this research, we study some elementary properties and provide some results on duality of the sequence spaces \( \mathcal{L}^r \). The main goal is to establish a decomposition theorem for the dual space \( (\mathcal{L}^r)^* \) of \( \mathcal{L}^r \) by a way analogous to the theorem of Dixmier mentioned in the first paragraph.
Chapter 2
Theoretical Background

In this chapter, we provide some theoretical background which is necessary for the research.
Throughout this thesis, we let \( \mathbb{C} \) and \( \mathbb{R} \) denote the sets of all complex numbers and real numbers respectively.

2.1 Banach Spaces

Definition 2.1.1. [9] Let \( X \) be a vector space over a scalar field \( K \) (\( K = \mathbb{R} \) or \( \mathbb{C} \)). A norm on \( X \) is a real-valued function \( \| \cdot \| \) on \( X \) satisfying the following properties:

(1) \( \| x \| \geq 0 \);
(2) \( \| x \| = 0 \) if and only if \( x = 0 \);
(3) \( \| \alpha x \| = |\alpha| \| x \| \);
(4) \( \| x + y \| \leq \| x \| + \| y \| \) (Triangle inequality),

where \( x \) and \( y \) are arbitrary vectors in \( X \) and \( \alpha \) is any scalar in \( K \). A normed space is a pair \( (X, \| \cdot \|) \) of a non-empty set \( X \) and a norm \( \| \cdot \| \) on \( X \). It may be sometimes written just \( X \) as a normed space by omitting the norm on \( X \).

Definition 2.1.2. [9] A sequence \( \{ x_n \}_{n=1}^{\infty} \) in a normed space \( X \) is said to converge or to be convergent if there is a point \( x \) in \( X \) satisfying the following property: for any \( \epsilon > 0 \), there is a positive integer \( N \) such that

\[
\| x - x_n \| < \epsilon \quad \text{for all} \quad n \geq N.
\]

In this situation, we write \( \lim_{n \to \infty} x_n = x \), or simply \( x_n \to x \) and call \( x \) the limit of \( \{ x_n \}_{n=1}^{\infty} \).

Definition 2.1.3. [9] A sequence \( \{ x_n \}_{n=1}^{\infty} \) in a normed space \( X \) is said to be bounded if there is a positive real number \( c \) such that \( \| x_n \| \leq c \) for all positive integer \( n \).
Definition 2.1.4. [9] A sequence \( \{x_n\}_{n=1}^{\infty} \) in a normed space \( X \) is said to be a *Cauchy sequence* in \( X \) if for any \( \epsilon > 0 \), there is a positive integer \( N \) such that 
\[
\|x_m - x_n\| < \epsilon
\]
for all \( m, n \geq N \). A normed space \( X \) is said to be a *Banach space* if it is *complete* under the metric \( d \) defined by \( d(x, y) = \|x - y\| \), that is, every Cauchy sequence converges to an element in \( X \).

Definition 2.1.5. [9] Let \( X \) and \( Y \) be vector spaces over the same scalar field. A function \( T : X \to Y \) is said to be a *linear operator* or *linear function* or *linear transformation* if
\[
T(\alpha x_1 + \beta x_2) = \alpha Tx_1 + \beta Tx_2
\]
for every \( x_1, x_2 \in X \) and any scalars \( \alpha \) and \( \beta \).

Definition 2.1.6. [9] Let \( X \) and \( Y \) be normed spaces over the same scalar field. A linear operator \( T : X \to Y \) is said to be *bounded* if \( T(B) \) is bounded for all bounded subsets \( B \) of \( X \).

Definition 2.1.7. Let \( T \) be a linear operator from a normed space \( X \) into a normed space \( Y \). Then the range of \( T \) is denoted by \( \text{ran} \ T \). We call the set \( \{x \in X : Tx = 0\} \) the *kernel* of \( T \) and denote by \( \text{ker} \ T \).

Theorem 2.1.8. [9] Let \( T : X \to Y \) be a linear operator from a normed space \( X \) into a normed space \( Y \). Then the following are equivalent.

1. \( T \) is bounded.
2. \( T \) is continuous.
3. There is a constant \( M > 0 \) such that \( \|Tx\| \leq M \|x\| \) for all \( x \in X \).

Let \( \mathcal{B}(X, Y) \) be the set of all bounded linear operators from a normed space \( X \) into a normed space \( Y \). We denote \( \mathcal{B}(X, X) \) by just \( \mathcal{B}(X) \).

Definition 2.1.9. [9] Let \( X \) and \( Y \) be normed spaces. For each \( T \) in \( \mathcal{B}(X, Y) \), the *norm* or *operator norm* \( \|T\| \) of \( T \) is the nonnegative real number \( \sup\{\|Tx\| : x \in X, \|x\| \leq 1\} \). The operator norm on \( \mathcal{B}(X, Y) \) is the map \( T \mapsto \|T\| \).

From Theorem 2.1.8, the following corollary is immediately obtained.

Corollary 2.1.10. [9] If \( T \) is a bounded linear operator from a normed space \( X \) into a normed space \( Y \), then \( \|Tx\| \leq \|T\| \|x\| \) for all \( x \) in \( X \). Furthermore, the number \( \|T\| \) is the smallest nonnegative real number \( M \) such that \( \|Tx\| \leq M \|x\| \) for all \( x \in X \).
Definition 2.1.11. [9] Let $T$ be a linear operator from a normed space $X$ onto a normed space $Y$. The operator $T$ is an isometric isomorphism if $\|T(x)\| = \|x\|$ whenever $x \in X$.

Notice that the condition $\|T(x)\| = \|x\|$ for all $x \in X$ implies $T$ is an one-to-one function.

Theorem 2.1.12. [9] If $X$ is a normed space and $Y$ is a Banach space, then the set $\mathcal{B}(X,Y)$ equipped with the operator norm is a Banach space.

Theorem 2.1.13. [9] (The Uniform Boundedness Principle) Let $\mathcal{F}$ be a nonempty family of bounded linear operators from a Banach space $X$ into a normed space $Y$. If $\sup \{\|Tx\| : T \in \mathcal{F}\}$ is finite for each $x$ in $X$, then $\sup\{\|T\| : T \in \mathcal{F}\}$ is finite.

Definition 2.1.14. [9] A normed space $X$ is said to be the direct sum of its two subspaces $Y$ and $Z$, written by $X = Y \oplus Z$, if each $x \in X$ has a unique representation of the form $x = y + z$, where $y \in Y$ and $z \in Z$. If, in addition, the condition $\|x\| = \|y\| + \|z\|$ is satisfied for all $x \in X$, we say specifically that $X$ is the $l^1$ direct-sum of $Y$ and $Z$ and write $X = Y \oplus_1 Z$ in this situation.

Theorem 2.1.15. [9] Let $X$ be a normed space and $Y$ and $Z$ be subspaces of $X$. Then $X = Y \oplus Z$ if and only if for $X \cap Y = \{0\}$ and for every $x$ in $X$, there are $y \in Y$ and $z \in Z$ such that $x = y + z$.

Definition 2.1.16. [9] A linear functional $f$ is a linear operator from a normed space $X$ into the scalar field $\mathbb{K}$, where $\mathbb{K}$ is regarded as a normed space under the usual norm on $\mathbb{K}$.

If $X$ is a normed space, then the set of all bounded linear functionals on $X$ is denoted by $X^\ast$. By Theorem 2.1.11, the normed space $X^\ast$ is immediately a Banach space.

Theorem 2.1.17. [9] (Hahn-Banach extension theorem) Let $X$ be a Banach space and $Y$ a closed subspace of $X$. If $f_0$ is a bounded linear functional on $Y$, then there is a unique bounded linear functional $f$ on $X$ such that $f(x) = f_0(x)$ for all $x \in Y$ and $\|f\| = \|f_0\|$.

Definition 2.1.18. [9] Let $X$ be a normed space and $Y$ a subspace of $X$. The annihilator of $Y$, denoted by $Y^\perp$, is the set $\{f \in X^\ast : f(x) = 0 \text{ for all } x \in Y\}$.

Theorem 2.1.19. [9] If $X$ is a normed space and $Y$ is a subspace of $X$, then $Y^\perp$ is a closed subspace of $X^\ast$. 
Definition 2.1.20. [9] Let $X$ and $Y$ be Banach spaces. A linear operator $T : X \to Y$ is compact if $\overline{T(B)}$ is compact for all bounded subset $B$ of $X$. The set of all compact operators from $X$ into $Y$ will be denoted by $\mathcal{K}(X,Y)$. For the case where $X = Y$, we write $\mathcal{K}(X)$ instead of $\mathcal{K}(X,Y)$.

Proposition 2.1.21. [9] Let $X$ and $Y$ be Banach spaces. Then the following hold.

1. $\mathcal{K}(X,Y) \subseteq \mathcal{B}(X,Y)$.
2. $\mathcal{K}(X,Y)$ is a closed subspace of $\mathcal{B}(X,Y)$.
3. If $X = Y$, then $\mathcal{K}(X)$ is an ideal of $\mathcal{B}(X)$.

Definition 2.1.22. [9] A linear operator $T$ from a Banach space $X$ into a Banach space $Y$ is said to be of finite rank if $T(X)$ is finite dimensional.

Theorem 2.1.23. [9] A finite rank operator from a Banach space $X$ into a Banach space $Y$ is bounded if and only if it is compact.

2.2 $l^p$ Spaces

Definition 2.2.1. [9] For $1 \leq p \leq \infty$ and a sequence $\{\lambda_k\}_{k=1}^{\infty}$ of complex numbers, the $p$-norm of $\{\lambda_k\}_{k=1}^{\infty}$ is defined by

$$
\|\{\lambda_k\}_{k=1}^{\infty}\|_p = \begin{cases} 
\left(\sum_{k=1}^{\infty} |\lambda_k|^p\right)^{1/p} & \text{if } 1 \leq p < \infty, \\
\sup\{|\lambda_k| : k = 1, 2, 3, \ldots\} & \text{if } p = \infty.
\end{cases}
$$

For each $1 \leq p < \infty$, let

$$
l^p = \left\{\{\lambda_k\}_{k=1}^{\infty} \subseteq \mathbb{C} : \sum_{k=1}^{\infty} |\lambda_k|^p < \infty\right\}
$$

and

$$
l^\infty = \left\{\{\lambda_k\}_{k=1}^{\infty} \subseteq \mathbb{C} : \sup\{|\lambda_k| : k = 1, 2, 3, \ldots\} < \infty\right\}.
$$

Theorem 2.2.2. [9] (Hölder’s inequality) For any $1 \leq p \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and sequences $x$ and $y$ of complex numbers, $\|xy\|_1 \leq \|x\|_p \|y\|_q$.

In particular, Hölder’s inequality is also called Cauchy-Schwartz’s inequality when $p = q = 2$. From Hölder’s inequality, the following Minkowski’s inequality is obtained.
Theorem 2.2.3. [9] (Minkowski’s inequality) For any $1 \leq p \leq \infty$ and sequences $x$ and $y$ of complex numbers, $\|x + y\|_p \leq \|x\|_p + \|y\|_p$.

Theorem 2.2.4. [9] For any $1 \leq p \leq \infty$, the set $l^p$ endowed with the $p$-norm $\|\cdot\|_p$ is a Banach space.

For $1 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, we define, for each $x = \{x_k\}_{k=1}^{\infty} \in l^q$, the function $f_x : l^p \to \mathbb{C}$ by

$$f_x(\{y_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} x_k y_k$$

for all $\{y_k\}_{k=1}^{\infty} \in l^p$.

By Hölder’s inequality, we have that the function $f_x$ is well-defined.

Theorem 2.2.5. [9] Let $1 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then $l^q$ is isometrically isomorphic to $(l^p)^*$ by the isomorphism defined by $x \mapsto f_x$.

The following result is closely related to the duality theorem stated above.

Theorem 2.2.6. [9] Let $1 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then a sequence $\{x_k\}_{k=1}^{\infty}$ of complex numbers belongs to $l^q$ if and only if $\{x_k y_k\}_{k=1}^{\infty}$ belongs to $l^1$ for all $\{y_k\}_{k=1}^{\infty}$ in $l^p$.

2.3 Hilbert Spaces

Definition 2.3.1. [4] Let $H$ be a vector space over a scalar field $K$ (which is either $\mathbb{R}$ or $\mathbb{C}$), a semi-inner product on $H$ is a function $\langle \cdot, \cdot \rangle : H \times H \to K$ having the following properties:

(i) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$;

(ii) $\langle x, x \rangle \geq 0$;

(iii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

If $\langle \cdot, \cdot \rangle$ has the following additional property:

(iv) if $\langle x, x \rangle = 0$, then $x = 0$,

we call $\langle \cdot, \cdot \rangle$ an inner product on $H$.

From (i), we have $\langle 0, y \rangle = \langle 0x, y \rangle = 0\langle x, y \rangle = 0$, and similarly, $\langle x, 0 \rangle = 0$. In particular, $\langle 0, 0 \rangle = 0$. Hence if $\langle \cdot, \cdot \rangle$ is an inner-product, then $\langle x, x \rangle = 0$ if and only if $x = 0$. If $\langle \cdot, \cdot \rangle$ is an inner-product on $H$, then

$$\|x\| = \langle x, x \rangle^{1/2}$$
defines a norm on \( \mathcal{H} \). We call a vector space \( \mathcal{H} \) equipped with an inner product on \( \mathcal{H} \) an *inner product space*. Every inner product space is a normed space under the norm defined by \( \|x\| = \langle x, x \rangle^{1/2} \). If \( \mathcal{H} \) equipped with the norm \( \| \cdot \| \) is a Banach space, we call \( \mathcal{H} \) a *Hilbert space*.

Let, in the sequel, \( \mathcal{H} \) and \( \mathcal{L} \) be Hilbert spaces.

**Definition 2.3.2.** [4] If \( f, g \in \mathcal{H} \), then \( f \) and \( g \) are *orthogonal* if \( \langle f, g \rangle = 0 \), in symbols, \( f \perp g \). If \( A, B \subseteq \mathcal{H} \), we say that \( A \) and \( B \) are *orthogonal* and write \( A \perp B \) provided \( f \perp g \) for every \( f \in A \) and \( g \in B \). If \( A \subseteq \mathcal{H} \) and \( f \in \mathcal{H} \) satisfying \( \{f\} \perp A \), then we write \( f \perp A \). If \( A \subseteq \mathcal{H} \), then the set \( A^\perp \) is defined by \( A^\perp = \{h \in \mathcal{H} : h \perp g \text{ for all } g \in A \} \).

**Definition 2.3.3.** [4] An *orthonormal set* in \( \mathcal{H} \) is a subset \( \mathcal{E} \) of \( \mathcal{H} \) having the following properties:

(i) for \( e \in \mathcal{E} \), \( \|e\| = 1 \);

(ii) if \( e_1, e_2 \in \mathcal{E} \) and \( e_1 \neq e_2 \), then \( e_1 \perp e_2 \).

An *orthonormal basis* for \( \mathcal{H} \) is a maximal orthonormal set.

**Proposition 2.3.4.** [4] If \( \mathcal{E} \) is an orthonormal set in \( \mathcal{H} \), then there is an orthonormal basis for \( \mathcal{H} \) that contains \( \mathcal{E} \).

**Theorem 2.3.5.** [4] If \( \mathcal{E} \) is an orthonormal set in \( \mathcal{H} \) and \( h \in \mathcal{H} \), then \( \{e \in \mathcal{E} : \langle h, e \rangle \neq 0 \} \) is countable.

**Theorem 2.3.6.** [4] Let \( \mathcal{E} \) be an orthonormal set in \( \mathcal{H} \). Then the following statements are equivalent.

1. \( \mathcal{E} \) is an orthonormal basis.
2. If \( h \in \mathcal{H} \) and \( h \perp \mathcal{E} \), then \( h = 0 \).
3. \( \bigvee \mathcal{E} = \mathcal{H} \), where \( \bigvee \mathcal{E} \) is the smallest closed subspace of \( \mathcal{H} \) containing \( \mathcal{E} \).
4. \( h = \sum \{\langle h, e \rangle e : e \in \mathcal{E} \} \) for all \( h \in \mathcal{H} \), where \( \sum \{\langle h, e \rangle e : e \in \mathcal{E} \} \) denotes the limit of the net \( \left\{ \sum_{e \in F} \langle h, e \rangle e : F \text{ is a finite subset of } \mathcal{E} \right\} \).

**Theorem 2.3.7.** Any two orthonormal bases of \( \mathcal{H} \) have the same cardinality.

**Definition 2.3.8.** [4] The *dimension* of \( \mathcal{H} \) is the cardinality of an orthonormal basis and is denoted by \( \dim \mathcal{H} \).
Definition 2.3.9. [4] A subset $D$ of $\mathcal{H}$ is said to be dense in $\mathcal{H}$ if $\overline{D} = \mathcal{H}$. $\mathcal{H}$ is said to be separable if it has a countable subset which is dense in $\mathcal{H}$.

Theorem 2.3.10. [4] Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then $\mathcal{H}$ is separable if and only if $\dim \mathcal{H} = \aleph_0$, where $\aleph_0$ is the cardinality of the set of all positive integers.

Definition 2.3.11. [4] A function $u : \mathcal{H} \times \mathcal{L} \to \mathbb{K}$ is a sesquilinear form if for $h, g$ in $\mathcal{H}$, $k, f$ in $\mathcal{L}$, and $\alpha, \beta$ in $\mathbb{K}$,

(i) $u(\alpha h + \beta g, k) = \alpha u(h, k) + \beta u(g, k)$;

(ii) $u(h, \alpha k + \beta f) = \overline{\alpha} u(h, k) + \overline{\beta} u(h, f)$.

Definition 2.3.12. [4] A sesquilinear form $u$ is bounded if there is a constant $M$ such that $|u(h, k)| \leq M \|h\| \|k\|$ for all $h$ in $\mathcal{H}$ and $k$ in $\mathcal{L}$. The constant $M$ is called a bound for $u$.

Theorem 2.3.13. [4] If $u : \mathcal{H} \times \mathcal{L} \to \mathbb{K}$ is a bounded sesquilinear form with a bound $M$, then there are unique operators $A$ in $\mathcal{B}(\mathcal{H}, \mathcal{L})$ and $B$ in $\mathcal{B}(\mathcal{L}, \mathcal{H})$ such that

$$u(h, k) = \langle Ah, k \rangle = \langle h, Bk \rangle$$

for all $h$ in $\mathcal{H}$ and $k$ in $\mathcal{L}$ and both $\|A\|$ and $\|B\|$ are not greater than $M$.

Definition 2.3.14. [4] If $A \in \mathcal{B}(\mathcal{H}, \mathcal{L})$, then the unique operator $B$ in $\mathcal{B}(\mathcal{L}, \mathcal{H})$ satisfying $\langle Ah, k \rangle = \langle h, Bk \rangle$ is called the adjoint of $A$ and is denoted by $A^*$.

Proposition 2.3.15. [4] If $A, B \in \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$ and $\alpha \in \mathbb{K}$, then the following hold.

1. $(\alpha A + B)^* = \overline{\alpha} A^* + B^*$.

2. $(AB)^* = B^* A^*$.

3. $A^{**} = (A^*)^* = A$.

4. If $A$ is invertible in $\mathcal{B}(\mathcal{H})$, then $(A^*)^{-1} = (A^{-1})^*$.

Theorem 2.3.16. [4] Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space with an orthonormal basis $\{e_n\}$. If $T \in \mathcal{B}(\mathcal{H})$ with the matrix representation $A = [a_{ji}]$ with respect to the basis $\{e_n\}$, then the matrix representation of $T^*$ with respect to $\{e_n\}$ is the matrix $[\overline{a_{ji}}]^t$. 
Definition 2.3.17. [4] A bounded linear operator $A$ on a Hilbert space $\mathcal{H}$ is said to be \textit{self-adjoint} if $A = A^\ast$.

Theorem 2.3.18. [4] If $T \in \mathcal{B}(\mathcal{H}, \mathcal{L})$, the following statements are equivalent.

(1) $T$ is compact.

(2) $T^\ast$ is compact.

(3) There is a sequence $\{T_n\}$ of operators of finite rank such that $\|T - T_n\| \to 0$.

Definition 2.3.19. [4] If $A \in \mathcal{B}(\mathcal{H})$, a scalar $\alpha$ is an \textit{eigenvalue} of $A$ if $\ker(A - \alpha I) \neq \{0\}$.

Definition 2.3.20. [4] If $T \in \mathcal{B}(\mathcal{H})$, then $T$ is \textit{positive} if $\langle Th, h \rangle \geq 0$ for all $h \in \mathcal{H}$.

In symbols, this is denoted by $T \geq 0$. Note that every positive operator on a complex Hilbert space is self-adjoint.

Theorem 2.3.21. [4] If $T$ is a positive compact operator on a Hilbert space $\mathcal{H}$, then there is a unique positive compact operator $A$ such that $A^2 = T$.

Definition 2.3.22. [4] If $T$ is a positive compact operator on a Hilbert space $\mathcal{H}$, then the unique positive compact operator $A$ such that $A^2 = T$ according to Theorem 2.3.21 is called the \textit{positive square root} of $T$ and denoted by $|T|$.

Definition 2.3.23. [4] A \textit{partial isometry} is a linear operator $W$ such that $\|Wh\| = \|h\|$ for all $h \in (\ker W)\perp$. The space $(\ker W)\perp$ is called the \textit{initial space} of $W$ and the space $\text{ran } W$ is called the \textit{final space} of $W$.

Theorem 2.3.24. [4] (Polar Decomposition) If $T \in \mathcal{B}(\mathcal{H})$, then there is a partial isometry $W$ with $(\ker T)\perp$ as its initial space and $\text{ran } T$ as its final space such that $T = W|T|$. Moreover, if $T = UP$ where $P \geq 0$ and $U$ is a partial isometry with $\ker U = \ker P$, then $P = |T|$ and $U = W$.

Theorem 2.3.25. [4] (Spectral Theorem) If $T$ is a compact self-adjoint operator on $\mathcal{H}$, then $T$ has only a countable number of distinct eigenvalues. If $\{\lambda_1, \lambda_2, \ldots\}$ are the distinct nonzero eigenvalue of $T$ with $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \ldots$, and $P_n$ is the projection of $\mathcal{H}$ onto $\ker(T - \lambda_n)$, then $P_nP_m = P_mP_n = 0$ if $n \neq m$, each $\lambda_n$ is an real, and

$$T = \sum_{n=1}^{\infty} \lambda_n P_n,$$

where the series converges to $T$ in the metric defined by the norm of $\mathcal{B}(\mathcal{H})$. 
Corollary 2.3.26. [4] With the notation of Spectral Theorem. One has the following.

1. \( \ker T = \text{span}(\bigcup_{n=1}^{\infty} P_n \mathcal{H}) = (\text{ran } T)^\perp; \)
2. each \( P_n \) has finite rank;
3. \( \|T\| = \sup\{|\lambda_n| : n \geq 1\} \) and \( \lambda_n \to 0 \) as \( n \to \infty \).

Let \( T \) be a compact self-adjoint operator on \( \mathcal{H} \). By Spectral Theorem, \( T \) has precisely finite or countable number of distinct eigenvalues. Let \( \{\lambda_n\}_{n=1}^{\infty} \) be the sequence of eigenvalues of \( T \) with \( |\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq ... \). For each \( n \), let \( N_n \) be the dimension of \( \ker(T - \lambda_n) \), and let \( \{\mu_n\}_{n=1}^{\infty} = \{\lambda_1, ..., \lambda_1, \lambda_2, ..., \lambda_2, ...\} \). If \( T \) has only \( k \) eigenvalues, then we let \( \mu_n = 0 \) for all \( n > N_1 + N_2 + ... + N_k \).

Corollary 2.3.27. [4] If \( T \) is a compact self-adjoint operator on \( \mathcal{H} \), then there is an orthonormal basis \( \{e_n\} \) for \( (\ker T)^\perp \) such that

\[
Th = \sum_{n=1}^{\infty} \mu_n \langle h, e_n \rangle e_n
\]

for all \( h \in \mathcal{H} \).

### 2.4 Schatten \( p \)-Classes

Let \( \mathcal{H} \) be an infinite dimensional separable Hilbert space and \( K \) a compact operator on \( \mathcal{H} \). Since \( 0 \leq \|K h\|^2 = \langle K h, K h \rangle = \langle K^* K h, h \rangle \) for all \( h \in \mathcal{H} \), it follows that \( K^* K \) is a positive compact operator. Whence, by Theorem 2.3.21, there is a unique positive compact operator \( |K| \) such that \( |K|^2 = K^* K \). Since \( |K| \) is positive, \( |K| \) is self-adjoint. Thus, by Corollary 2.3.27, we have \( \|K\| = \sum_{n=1}^{\infty} \mu_n \langle h, e_n \rangle e_n \) for all \( h \in \mathcal{H} \), where \( \{\mu_n\}_{n=1}^{\infty} \) is the sequence of eigenvalues of \( |K| \) and \( \{e_n\}_{n=1}^{\infty} \) is an orthonormal basis for \( (\ker |K|)^\perp \). Notice that ker \( K = \ker |K| \) due to the fact that \( \|K h\|^2 = \langle K h, K h \rangle = \langle h, K^* K h \rangle = \langle h, |K|^2 h \rangle = \| |K|h \| \cdot |K| h \| = \| |K|h \|^2 \) for all \( h \in \mathcal{H} \). We call the sequence \( \{\mu_n\}_{n=1}^{\infty} \) the sequence of singular values of the compact operator \( K \) and denote \( \mu_n \) by \( s_n(K) \) for all \( n \). By Corollary 2.3.26, we have \( \|K\| = s_1(K) \geq s_2(K) \geq ... \geq 0 \) and \( \lim_{n \to \infty} s_n(K) = 0 \).

For \( 1 \leq p \leq \infty \), let

\[
\mathcal{C}^p = \{ K \in \mathcal{K}(\mathcal{H}) : \{s_k(K)\}_{k=1}^{\infty} \in l^p \}.
\]

The set \( \mathcal{C}^p \) is called the Schatten \( p \)-class. We define, for \( 1 \leq p < \infty \), the norm \( \|\cdot\|_p \) on \( \mathcal{C}^p \) by

\[
\|K\|_p = \left( \sum_{n=1}^{\infty} s_n(K)^p \right)^{1/p}.
\]

For \( p = \infty \), we define \( \|K\|_\infty = \sup s_n(K) \). It is obvious that for any compact operator \( K \) on \( \mathcal{H} \), \( \|K\|_\infty = s_1(K) = \|K\| \). Thus \( \mathcal{C}^\infty = \mathcal{K}(\mathcal{H}) \).
Theorem 2.4.1. [5] If $K \in C^1$, then for each orthonormal basis $\{e_n\}$ of $\mathcal{H}$ the sum

$$\sum_{n=1}^{\infty} \langle Ke_n, e_n \rangle$$

is absolutely convergent and

$$\sum_{n=1}^{\infty} \langle Ke_n, e_n \rangle = \sum_{n=1}^{\infty} s_n(K) \langle Ue_n, e_n \rangle,$$

where $U$ is the unique partial isometry such that $K = U|K|$. 

Definition 2.4.2. [5] For each $K \in C^1$, the number

$$\sum_{n=1}^{\infty} \langle Ke_n, e_n \rangle,$$

where $\{e_n\}$ is an orthonormal basis of $\mathcal{H}$, is called the trace of $K$. 

Remark 2.4.3. [5] If $\{e_n\}$ is an ordered orthonormal basis of $\mathcal{H}$ and $K \in C^1$ with the matrix representation $A$ with respect to $\{e_n\}$, then the trace of $K$ is exactly the sum of all entries in the main diagonal of $A$.

Theorem 2.4.4. [5] For each $1 < p \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, $(C^p)^* \cong C^q$.

Theorem 2.4.5. [5] $(C^1)^* \cong B(\mathcal{H})$. 
Chapter 3

Duality of Sequence Spaces of Infinite Matrices

3.1 Basic Results

Recall that, for any $1 \leq r < \infty$, the set $\mathcal{L}^r$ of sequences of infinite matrices is defined by

$$\mathcal{L}^r = \left\{ \left\{ \left[ a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty} \subset M_\infty : \sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^r \in B(l^2) \right\}.$$ 

It is clear that if $\{A_k\}_{k=1}^{\infty} \subset \mathcal{L}^r$, then $A_k$ is necessarily a member of the absolute Schur algebra $S^r_{2,2}(\mathbb{C})$ for all $k$.

The following theorem was first stated in [3] by A. Charearnpol. It is a generalization of the characterization of the sequence spaces $O_b$ provided by J. Rakbud and S.-C. Ong in [11].

**Theorem 3.1.1.** Let $\left\{ \left[ a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty}$ be a sequence in $B(l^2)$ with $a_{ji}^{(k)} \geq 0$ for all $i, j, k$.

1. The sequence $\left\{ \sum_{n=1}^{\infty} \left[ a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty}$ is bounded in $B(l^2)$ if and only if $\left[ \sum_{k=1}^{\infty} a_{ji}^{(k)} \right] \in B(l^2)$.

2. If $\left[ \sum_{k=1}^{\infty} a_{ji}^{(k)} \right] \in B(l^2)$, then $\left\| \sum_{k=1}^{\infty} a_{ji}^{(k)} \right\| = \sup_{n} \left\| \sum_{k=1}^{n} a_{ji}^{(k)} \right\|$.

From the above theorem, the following characterization of the set $\mathcal{L}^r$ is immediately obtained.

**Corollary 3.1.2.** Let $\{A_k\}_{k=1}^{\infty}$ be a sequence in $M_\infty$ and $1 \leq r < \infty$. Then the following are equivalent:

1. $\{A_k\}_{k=1}^{\infty}$ belongs to $\mathcal{L}^r$;
(2) $A_k \in S_{2,2}(\mathbb{C})$ for all $k$ and the sequence $\left\{ \sum_{k=1}^{n} A_k^r \right\}_{n=1}^{\infty}$ is bounded in $B(l^2)$;

(3) the sequence $\left\{ \left\| \sum_{k=1}^{n} A_k^r \right\| \right\}_{n=1}^{\infty}$ is bounded.

For any sequence $\left\{ a_{ji}^{(k)} \right\}_{k=1}^{\infty}$ in $\mathcal{M}_\infty$ and $1 \leq r < \infty$, we define

$$\left\| \left\{ a_{ji}^{(k)} \right\}_{k=1}^{\infty} \right\|_r = \left\{ \left[ \sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^r \right]^{1/r} \right\}_{k=1}^{\infty}$$

if $\left\{ a_{ji}^{(k)} \right\}_{k=1}^{\infty} \in \mathcal{L}^r$, otherwise.

The following Hölder-type inequality was first established in [2] by Chaisuriya and Ong. It is useful for the research.

**Theorem 3.1.3.** (Hölder-type inequality) For any $A, B \in \mathcal{M}_\infty$ and $1 < r < \infty$ with $\frac{1}{r} + \frac{1}{r^*} = 1$,

$$\left\| (A \cdot B)^{[1]} \right\| \leq \left\| A \right\|^{1/r} \left\| B \right\|^{1/r^*}$$

under the conventions that $\infty \cdot 0 = 0 \cdot \infty = 0$, $\infty \cdot \alpha = \alpha \cdot \infty = \infty$ for all positive real number $\alpha$ and $\infty \cdot \infty = \infty$.

The Hölder and Minkowski-type inequalities below are extensions of the ones in [11].

**Theorem 3.1.4.** (Hölder-type inequality for sequences of matrices) For any sequences $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ in $\mathcal{M}_\infty$,

$$\left\| \left\{ A_k \cdot B_k \right\}_{k=1}^{\infty} \right\|_r \leq \left\{ \left\| \{ A_k \}_{k=1}^{\infty} \right\|_r \right\} \left\{ \left\| \{ B_k \}_{k=1}^{\infty} \right\|_{r^*} \right\},$$

where $1 < r < \infty$ with $\frac{1}{r} + \frac{1}{r^*} = 1$, under the same convention as in Theorem 3.1.3.

**Proof.** Let $\{A_k = \left[ a_{ji}^{(k)} \right]_{k=1}^{\infty}\}$ and $\{B_k = \left[ b_{ji}^{(k)} \right]_{k=1}^{\infty}\}$ be sequences in $\mathcal{M}_\infty$. If either $\left\{ \left\{ A_k \right\}_{k=1}^{\infty} \right\} \left\|_r$ or $\left\{ \left\{ B_k \right\}_{k=1}^{\infty} \right\} \left\|_{r^*}$ is infinite, then we are done. Suppose that both $\left\{ \left\{ A_k \right\}_{k=1}^{\infty} \right\} \left\|_r$ and $\left\{ \left\{ B_k \right\}_{k=1}^{\infty} \right\} \left\|_{r^*}$ are finite. Then $\left[ \sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^r \right]$ and $\left[ \sum_{k=1}^{\infty} \left| b_{ji}^{(k)} \right|^{r^*} \right]$ belong to $\mathcal{B}(l^2)$.

Thus, by Hölder’s inequality, we have for each $i, j$ that

$$\sum_{k=1}^{\infty} \left| a_{ji}^{(k)} b_{ji}^{(k)} \right| \leq \left( \sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^r \right)^{1/r} \left( \sum_{k=1}^{\infty} \left| b_{ji}^{(k)} \right|^{r^*} \right)^{1/r^*} < \infty.$$
Hence the matrix \( \left[ \sum_{k=1}^{\infty} |a_{ji}^{(k)} b_{ji}^{(k)}| \right] \in \mathcal{M}_\infty \). We want to show that \( \left[ \sum_{k=1}^{\infty} |a_{ji}^{(k)} b_{ji}^{(k)}| \right] \in \mathcal{B}(l^2) \) and \( \|A_k \cdot B_k\|_{k=1}^\infty \leq \|A_k\|_{k=1}^\infty \|B_k\|_{k=1}^\infty \) for all non-negative real number \( \alpha \) and \( \infty + \alpha = \infty + \infty = \infty \). By the Hölder-type inequality, we have

\[
\left\| \sum_{k=1}^{\infty} \left| a_{ji}^{(k)} b_{ji}^{(k)} \right| \right\| \leq \left\| \left( \sum_{k=1}^{\infty} |a_{ji}^{(k)}|^r \right)^{1/r} \right\| \left\| \left( \sum_{k=1}^{\infty} |b_{ji}^{(k)}|^r \right)^{1/r} \right\| \leq \left\| \sum_{k=1}^{\infty} |a_{ji}^{(k)}|^r \right\|^{1/r} \left\| \sum_{k=1}^{\infty} |b_{ji}^{(k)}|^r \right\|^{1/r} \, .
\]

This implies that \( \left[ \sum_{k=1}^{\infty} |a_{ji}^{(k)} b_{ji}^{(k)}| \right] \in \mathcal{B}(l^2) \), which is equivalent to that \( \{A_k \cdot B_k\}_{k=1}^\infty \in \mathcal{L}^1 \), and \( \|A_k \cdot B_k\|_{k=1}^\infty \leq \|A_k\|_{k=1}^\infty \|B_k\|_{k=1}^\infty \| \cdot \|_{r} \).

**Theorem 3.1.5.** (Minkowski-type inequality for sequences of matrices) For any sequences \( \{A_k\}_{k=1}^\infty \) and \( \{B_k\}_{k=1}^\infty \) in \( \mathcal{M}_\infty \) and \( 1 \leq r < \infty \),

\[
\|\{A_k + B_k\}_{k=1}^\infty \|_r \leq \|\{A_k\}_{k=1}^\infty \|_r + \|\{B_k\}_{k=1}^\infty \|_r
\]

under the conventions that \( \infty + \alpha = \infty + \infty = \infty \) for all non-negative real number \( \alpha \) and \( \infty + \infty = \infty \).

**Proof.** For the case where either \( \|\{A_k\}_{k=1}^\infty \|_r = \infty \) or \( \|\{B_k\}_{k=1}^\infty \|_r = \infty \), there is nothing to prove. Suppose that both \( \|\{A_k\}_{k=1}^\infty \|_r \) and \( \|\{B_k\}_{k=1}^\infty \|_r \) are finite. We assume first that \( 1 < r < \infty \). Then by the Hölder-type inequality for sequences of matrices, we have for each positive integer \( n \) that

\[
\left\| \sum_{k=1}^{n} (A_k + B_k)^r \right\| = \left\| \sum_{k=1}^{n} (A_k + B_k)^{[1]} \cdot (A_k + B_k)^{[r-1]} \right\|
\]

\[
\leq \left\| \sum_{k=1}^{n} \left( A_k^{[1]} + B_k^{[1]} \right) \cdot (A_k + B_k)^{[r-1]} \right\|
\]

\[
= \left\| \sum_{k=1}^{n} A_k^{[1]} \cdot (A_k + B_k)^{[r-1]} + \sum_{k=1}^{n} B_k^{[1]} \cdot (A_k + B_k)^{[r-1]} \right\|
\]

\[
\leq \left\| \sum_{k=1}^{n} A_k^{[1]} \cdot (A_k + B_k)^{[r-1]} \right\| + \left\| \sum_{k=1}^{n} B_k^{[1]} \cdot (A_k + B_k)^{[r-1]} \right\|
\]

\[
\frac{1}{r} + \frac{1}{r^*} = 1, \quad \text{which implies that}
\]

\[
\left\| \sum_{k=1}^{n} (A_k + B_k)^{[r]} \right\|^{1/r} \leq \left\| \sum_{k=1}^{n} A_k^{[r]} \right\|^{1/r} + \left\| \sum_{k=1}^{n} B_k^{[r]} \right\|^{1/r} \leq \| \{ A_k \}_{k=1}^{\infty} \|_r + \| \{ B_k \}_{k=1}^{\infty} \|_r.
\]

For the case where \( r = 1 \), we easily have for each positive integer \( n \) that

\[
\left\| \sum_{k=1}^{n} (A_k + B_k)^{[1]} \right\| \leq \left\| \sum_{k=1}^{n} A_k^{[1]} \right\| + \left\| \sum_{k=1}^{n} B_k^{[1]} \right\| \leq \| \{ A_k \}_{k=1}^{\infty} \|_1 + \| \{ B_k \}_{k=1}^{\infty} \|_1
\]
as well. Thus, by Corollary 3.1.2, the sequence \( \{ A_k + B_k \}_{k=1}^{\infty} \) belongs to \( \mathcal{L}^r \) and

\[
\| \{ A_k + B_k \}_{k=1}^{\infty} \|_r \leq \| \{ A_k \}_{k=1}^{\infty} \|_r + \| \{ B_k \}_{k=1}^{\infty} \|_r
\]
for all \( 1 \leq r < \infty \). The proof is complete. \( \square \)

The following lemma was first stated and proved in [10]. It is a beautiful consequence of the Hölder-type inequality.

**Lemma 3.1.6.** For any \( 1 \leq r < \infty \) and matrices \( A \) and \( B \) in \( S_{2,2}^r(\mathbb{C}) \),

\[
\| A^{[r]} - B^{[r]} \| \leq (\| A \|_{2,2,r} + \| B \|_{2,2,r}) \| A - B \|_{2,2,r}.
\]

The proposition below was first stated and proved in [10] as well. We can see that it follows easily from the lemma above.

**Proposition 3.1.7.** For any \( 1 \leq r < \infty \), the map \( A \mapsto A^{[r]} \) from \( S_{2,2}^r(\mathbb{C}) \) into \( B(l^2) \) is continuous.

**Theorem 3.1.8.** For each \( 1 \leq r < \infty \), the set \( \mathcal{L}^r \) equipped with the norm \( \| \cdot \|_r \) is a Banach space.
Proof. From Minkowski’s inequality for sequences of matrices, we have that the set \( \mathcal{L}^r \) endowed with the norm \( \| \cdot \| \), is a normed space. To see that it is a Banach space, let \( \{ A_n = \{ A_k^{(n)} \}_{k=1}^{\infty} \}_{n=1}^{\infty} \) be a Cauchy sequence in \( \mathcal{L}^r \). As we have for each \( k \) that

\[
\left\| A_k^{(n)} - A_k^{(m)} \right\|_{\mathcal{L}^r} \leq \left\| A_n - A_m \right\|_{\mathcal{L}^r},
\]

it follows that the sequence \( \{ A_k^{(n)} \}_{n=1}^{\infty} \) is a Cauchy sequence in \( S_{2,2}(\mathbb{C}) \) for all \( k \). So, for each \( k \), we obtain by the completeness of \( S_{2,2}(\mathbb{C}) \) that there exists an \( A_k \) such that \( A_k^{(n)} \to A_k \). Let \( A = \{ A_k \}_{k=1}^{\infty} \). We claim that \( A \in \mathcal{L}^r \) and \( A_n \to A \). To prove these, let \( \epsilon > 0 \) be given. Then there is a positive integer \( N \) such that for each positive integer \( K \),

\[
\left\| \sum_{k=1}^{K} (A_k^{(n)} - A_k^{(m)})^{[r]} \right\|^{1/r} \leq \left\| A_n - A_m \right\|_{\mathcal{L}^r} < \frac{\epsilon}{2} \quad \text{for all } n, m \geq N. \tag{*}
\]

Since \( A_k^{(m)} \to A_k \) in \( S_{2,2}(\mathbb{C}) \) for all \( k \), it follows for each fixed \( n \) that \( A_k^{(n)} - A_k^{(m)} \to A_k^{(n)} - A_k \) in \( S_{2,2}(\mathbb{C}) \) for all \( k \). Thus, by Proposition 3.1.7, we obtain for each fixed \( n \) that \( \left( A_k^{(n)} - A_k^{(m)} \right)^{[r]} \to \left( A_k^{(n)} - A_k \right)^{[r]} \) in \( B(l^2) \) for all \( k \). From this we have for each fixed \( n \) and \( K \) that

\[
\left\| \sum_{k=1}^{K} (A_k^{(n)} - A_k)^{[r]} \right\|^{1/r} \leq \frac{\epsilon}{2} \quad \text{for all } K \geq 1.
\]

Therefore, by Theorem 3.1.1,

\[
\left\| A_n - A \right\|_{\mathcal{L}^r} = \sup_K \left\| \sum_{k=1}^{K} (A_k^{(n)} - A_k)^{[r]} \right\|^{1/r} < \epsilon \quad \text{for all } n \geq N. \tag{**}
\]

The inequality (**) yields that \( A_n - A \) belongs to \( \mathcal{L}^r \), which implies that \( A = A_N - (A_N - A) \) is an element of \( \mathcal{L}^r \). Consequently, by (**) again, we get \( A_n \to A \). \( \square \)

### 3.2 Duality

In this section, we study the duality of the sequence spaces \( \mathcal{L}^r \). The aim is to decompose the dual space \( (\mathcal{L}^r)^* \) of \( \mathcal{L}^r \) as an \( l^1 \) direct-sum of its two closed subspaces. Before getting the results, we need some notational conventions.

For any \( z \in \mathbb{C} \), we define the function \( \text{sgn}(\cdot) \) on \( \mathbb{C} \) by

\[
\text{sgn}(z) = \begin{cases} 
\frac{z}{|z|} & \text{if } z \neq 0, \\
1 & \text{if } z = 0.
\end{cases}
\]
For any sequences \( A = \{A_k\}_{k=1}^{\infty} \) and \( B = \{B_k\}_{k=1}^{\infty} \) in \( \mathcal{M}_\infty \) and any positive integer \( n \), we let \( A \cdot B = \{A_k \cdot B_k\}_{k=1}^{\infty} \), \( A_{n_j} = \{(A_k)_{n_j}\}_{k=1}^{\infty} \), \( \mathcal{A}_r = \{(A_k)_r\}_{k=1}^{\infty} \), and \( \mathcal{A}_n = \{A_1, A_2, ..., A_n, 0, 0, ...\} \). It is clear that \( (A_K)_{n_j} = (A_{n_j})_K \) for all positive integers \( n \) and \( K \). Notice that for each \( 1 \leq r < \infty \), if \( A = \{A_k = [a_{ji}^{(k)}]\}_{k=1}^{\infty} \in \mathcal{L}_r \), then each of the following holds true:

\[
(i) \|A_{n_j}\|_r = \left(\sum_{k=1}^{\infty} |a_{ji}^{(k)}|^r\right)^{1/r},
\]

\[
(ii) \|A_K\|_r = \left(\sum_{k=1}^{K} |a_{ji}^{(k)}|^r\right)^{1/r}
\]

\[
(iii) \|A_{n_j} - A\|_r = \left(\sum_{k=1}^{\infty} |(A_k)_{n_j} - A_k|^r\right)^{1/r} = \left(\sum_{k=1}^{\infty} |a_{ji}^{(k)}|^r\right)^{1/r} - \left(\sum_{k=1}^{\infty} |a_{ji}^{(k)}|^r\right)^{1/r}
\]

for all \( n \) and \( K \). The first and second equations imply that \( \|A\|_r = \sup_n \|A_{n_j}\|_r \) and \( \|A\|_r = \sup_n \|A_K\|_r \) respectively. And the last one implies that the matrix \( \left[\sum_{k=1}^{\infty} |a_{ji}^{(k)}|^r\right] \) is compact if and only if \( \|A_{n_j} - A\|_r \to 0 \). For each \( A = [a_{ji}] \) in \( \mathcal{M}_\infty \) and positive integer \( k \), let \( \sum A = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ji} \) if the series converges, let

\[\text{sgn} A = \text{sgn}(a_{ji})\text{, and let } s(A;k) \text{ be the sequence whose } k\text{-th term is the matrix } A \text{ and all other terms are } 0.\]

Finally, for any \( \lambda \in \mathbb{C} \) and pair \( (j, i) \) of positive integers, let \( E(\lambda; (j, i)) \) be the matrix whose \( (j, i) \)-th entry is the number \( \lambda \) and all other entries are 0.

On the classical sequence spaces \( l^p \), there is a result closely related to their duality as follows: for \( 1 \leq p < \infty \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), a sequence \( x \) belongs to \( l^q \) if and only if \( \langle x, y \rangle \) “Schur multiplies” every \( y \) in \( l^q \) into \( l^1 \). An analogue of this result is also obtained for our sequence spaces \( \mathcal{L}_r \) of infinite matrices.

**Theorem 3.2.1.** Let \( 1 < r < \infty \) with \( \frac{1}{r} + \frac{1}{r^*} = 1 \).

1. \( \{A_k\}_{k=1}^{\infty} \in \mathcal{L}_r \) if and only if \( \{A_k \cdot B_k\}_{k=1}^{\infty} \in \mathcal{L}_1 \) for all \( \{B_k\}_{k=1}^{\infty} \in \mathcal{L}_r \).
2. If \( \{A_k\}_{k=1}^{\infty} \in \mathcal{L}_r \), then

\[
\|\{A_k\}_{k=1}^{\infty}\|_r = \sup\{\|\{A_k \cdot B_k\}_{k=1}^{\infty}\|_1 : \{B_k\}_{k=1}^{\infty} \in \mathcal{L}_r, \|\{B_k\}_{k=1}^{\infty}\|_r \leq 1\}.
\]

**Proof.** (1). Let \( A = \{A_k = [a_{ji}^{(k)}]\}_{k=1}^{\infty} \) be a sequence in \( \mathcal{M}_\infty \). Suppose that \( \{A_k \cdot B_k\}_{k=1}^{\infty} \in \mathcal{L}_1 \) for all \( \{B_k\}_{k=1}^{\infty} \in \mathcal{L}_r \). We want to show that \( \{A_k\}_{k=1}^{\infty} \in \mathcal{L}_r \). By the assumption, a map \( \Psi : \mathcal{L}_r \to \mathcal{L}_1 \) can be defined as follows: \( \Psi(\{B_k\}_{k=1}^{\infty}) = \{A_k \cdot B_k\}_{k=1}^{\infty} \) for all \( \{B_k\}_{k=1}^{\infty} \in \mathcal{L}_r \). For any positive integer \( n \), let \( \Psi_n : \mathcal{L}_r \to \mathcal{L}_1 \) be
defined by $\Psi_n(\{B_k\}_{k=1}^\infty) = A_n \cdot B$ for all $B = \{B_k\}_{k=1}^\infty \in \mathcal{L}^r$. Then by the Hőlder-type inequality for sequences of matrices, we have for every $B = \{B_k\}_{k=1}^\infty \in \mathcal{L}^r$ that

$$
\|\Psi_n(\{B_k\}_{k=1}^\infty)\|_1 = \|A_n \cdot B\|_1 \leq \|A_n\|_r \cdot \|B\|_r.
$$

So the operator $\Psi_n$ is bounded for all $n$. For each $B_k = [b_{ji}^{(k)}]_{k=1}^\infty \in \mathcal{L}^r$, we have

$$
\|\Psi_n(\{B_k\}_{k=1}^\infty)\|_1 = \left\| \sum_{k=1}^n a_{ji}^{(k)} b_{ji}^{(k)} \right\|_1 \leq \left\| \sum_{k=1}^\infty a_{ji}^{(k)} b_{ji}^{(k)} \right\|_1 = \|\{A_k \cdot B_k\}_{k=1}^\infty\|_1 \text{ for all } n.
$$

Hence, by the uniform boundedness principle, the set $\{\|\Psi_n\| : n = 1, 2, 3, \ldots\}$ is bounded. For every $B = \{B_k\}_{k=1}^\infty \in \mathcal{L}^r$ with $\|B\|_r \leq 1$, we have by Theorem 3.1.1 that

$$
\|\Psi(B)\|_1 \leq \sup_n \left\| \sum_{k=1}^n (A_k \cdot B_k)^{(n)} \right\| = \sup_n \|A_n \cdot B\|_1
$$

Thus, by the boundedness of the set $\{\|\Psi_n\| : n = 1, 2, 3, \ldots\}$, the operator $\Psi$ is bounded. Next, let $D = \{A_k^{[r-1]}\}_{k=1}^\infty$. Then $(D_{n, K}) \in \mathcal{L}^r$ for all $n, K$. Thus

$$
\left\| \left( \sum_{k=1}^K A_k^{[r]} \right)_{n, K} \right\|_1 = \left\| \sum_{k=1}^K (A_k^{[r-1]})_{n, K} \right\| = \left\| \sum_{k=1}^K (A_k^{[r-1]} \cdot (A_k^{[r-1]})_{n, K} \right\|_1
$$

$$
= \left\| \Psi \cdot (D_{n, K}) \right\|_1 \leq \left\| \Psi \right\| \left\| \sum_{k=1}^K (A_k^{[r-1]} \cdot (A_k^{[r-1]})_{n, K} \right\|^{1/r}_1
$$

$$
= \left\| \Psi \right\| \left\| \left( \sum_{k=1}^K A_k^{[r]} \right)_{n, K} \right\|^{1/r}_1 \text{ for all } n, K.
$$

It follows that

$$
\left\| \left( \sum_{k=1}^K A_k^{[r]} \right)_{n, K} \right\|_1 \leq \|\Psi\|^{r^*} \text{ for all } n, K.
$$

Whence, by Theorem 1.1(2), we obtain for each $K$ that $\sum_{k=1}^K A_k^{[r^*]} \in B(l^2)$ and by Theorem 1.1(3),

$$
\left\| \sum_{k=1}^K A_k^{[r^*]} \right\|_1 = \sup_n \left\| \left( \sum_{k=1}^K A_k^{[r^*]} \right)_{n, K} \right\|_1 \leq \|\Psi\|^{r^*}.
$$

Therefore, by Corollary 3.1.2, the sequence $A$ belongs to $\mathcal{L}^{r^{**}}$. Conversely, suppose that $A \in \mathcal{L}^{r^{**}}$. Then for any $B_k = \{B_k\}_{k=1}^\infty \in \mathcal{L}^r$, we have by the Hőlder-type inequality for sequences of matrices that $\{A_k \cdot B_k\}_{k=1}^\infty \in \mathcal{L}^1$. 
(2). Suppose that \( \{A_k\}_{k=1}^{\infty} \in \mathcal{L}^r \). Then by (1), the linear operator \( \Psi : \mathcal{L}^r \to \mathcal{L}^1 \) defined by \( \{B_k\}_{k=1}^{\infty} \mapsto \{A_k \cdot B_k\}_{k=1}^{\infty} \) is well-defined, and by the Hölder-type inequality for sequences of matrices, it is obvious that \( \|\Psi\| \leq \|\{A_k\}_{k=1}^{\infty}\|_{r^*} \).

By the same argument as given in the proof of (1) (see the argument to obtain the inequality \((\star)\)), we have

\[
\left\| \sum_{k=1}^{n} A_k^{[r^*]} \right\|^{1/r^*} \leq \|\Psi\| \text{ for all } n.
\]

It follows from Theorem 3.1.1 that \( \|\{A_k\}_{k=1}^{\infty}\|_{r^*} \leq \|\Psi\| \). Consequently, we obtain

\[
\|\{A_k\}_{k=1}^{\infty}\|_{r} = \|\Psi\| = \sup\{\|\{A_k \cdot B_k\}_{k=1}^{\infty}\|_{1} : \{B_k\}_{k=1}^{\infty} \in \mathcal{L}^r, \|\{B_k\}_{k=1}^{\infty}\|_{r} \leq 1\}
\]

as required. The proof is complete.

For each \( 1 \leq r < \infty \), let

\[
\mathcal{L}_r^\kappa = \left\{ \left\{ a_{ki}^{(k)} \right\}_{k=1}^{\infty} \subseteq M_{\infty} : \left[ \sum_{k=1}^{\infty} |a_{ki}^{(k)}|^r \right] \in \mathcal{K}(l^2) \right\}.
\]

The following results on the sets \( \mathcal{L}_r^\kappa \) are evident.

(i) \( \mathcal{L}_r^\kappa \not\subset \mathcal{L}^r \).

(ii) A sequence \( A \) in \( M_{\infty} \) belongs to \( \mathcal{L}_r^\kappa \) if and only if \( \|A - A_n\|_r \to 0 \).

(iii) If a sequence \( A \) belongs to \( \mathcal{L}_r^\kappa \), then \( A - A_n \) belongs to \( \mathcal{L}_r^\kappa \) for all \( n \).

The following theorem is a more general version of the characterization of the sequence space \( \mathcal{O}_\kappa \) provided by Rakbud et al. in [11].

**Theorem 3.2.2.** Let \( \{A_k = \left[ a_{ji}^{(k)} \right]_{k=1}^{\infty} \}_{i,j} \) be a sequence in \( M_{\infty} \) with \( a_{ji}^{(k)} \geq 0 \) for all \( i,j,k \). Then \( \left[ \sum_{k=1}^{\infty} a_{ji}^{(k)} \right] \in \mathcal{K}(l^2) \) if and only if \( A_k \in \mathcal{K}(l^2) \) for all \( k \) and the sequence \( \left\{ \sum_{k=1}^{n} A_k \right\}_{k=1}^{\infty} \) converges in \( \mathcal{B}(l^2) \).

**Proof.** Suppose that the matrix \( A = \left[ \sum_{k=1}^{\infty} a_{ji}^{(k)} \right] \) is compact. Then for each \( k \), we have by Theorem 1.1(1) that \( A_k \in \mathcal{B}(l^2) \) and

\[
\|A_k - (A_k)_n\| \leq \|A - A_n\| \to 0.
\]

Thus \( A_k \) is compact for all \( k \). To see that the sequence \( \left\{ \sum_{k=1}^{n} A_k \right\}_{k=1}^{\infty} \) converges in \( \mathcal{B}(l^2) \), let \( \epsilon > 0 \) be given. Then by the compactness of the matrix \( A \), there exists
a positive integer \( N \) such that \( \|A_{N_j} - A\| < \frac{\epsilon}{3} \). As the series \( \sum_{k=1}^{\infty} a_{ji}^{(k)} \) converges for all \( 1 \leq j, i \leq N \), there is a positive integer \( K_0 \) such that for each \( 1 \leq j, i \leq N \),
\[ \sum_{k=K}^{\infty} a_{ji}^{(k)} < \frac{\epsilon}{3N^{3/2}} \] for all \( K \geq K_0 \). Hence for each \( K \geq K_0 \),
\[ \left\| \sum_{k=1}^{K} A_k - A \right\| \leq \left\| A_{N_j} - \left( \sum_{k=1}^{K} A_k \right) \right\| + \left\| \left( \sum_{k=1}^{K} A_k \right) - A \right\| = \sum_{j=1}^{\infty} \left( \sum_{k=1}^{K} a_{ji}^{(k)} \right)^2 + 2\|A_{N_j} - A\| < \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon. \]
This yields \( \sum_{k=1}^{\infty} A_k = A \) in \( B(l^2) \). Conversely, suppose that \( A_k \) is compact for all \( k \) and that the sequence \( \left\{ \sum_{k=1}^{n} A_k \right\}_{n=1}^{\infty} \) converges in \( B(l^2) \). It is clear that \( \sum_{k=1}^{\infty} A_k = \left[ \sum_{k=1}^{\infty} a_{ji}^{(k)} \right] \).
Since \( K(l^2) \) is closed in \( B(l^2) \), it follows that \( \sum_{k=1}^{\infty} A_k \) is compact. Thus we obtain that
\[ \sum_{k=1}^{\infty} a_{ji}^{(k)} \] is compact as required. \( \square \)

The following characterization of the set \( \mathcal{L}_{r}^{\kappa} \) is an immediate consequence of Theorem 3.2.2 above.

**Corollary 3.2.3.** Let \( \{A_k\}_{k=1}^{\infty} \) be a sequence in \( \mathcal{M}_\infty \) and \( 1 \leq r < \infty \). Then \( \{A_k\}_{k=1}^{\infty} \in \mathcal{L}_{r}^{\kappa} \) if and only if \( A_k^{[r]} \) is compact for all \( k \) and the sequence \( \left\{ \sum_{k=1}^{n} A_k^{[r]} \right\}_{k=1}^{\infty} \) converges in \( B(l^2) \).

**Theorem 3.2.4.** For each \( 1 \leq r < \infty \), the set \( \mathcal{L}_{r}^{\kappa} \) is a Banach subspace of \( \mathcal{L}^{r} \).

**Proof.** For any matrix \( A \in \mathcal{M}_\infty \) and positive integer \( n \), we let here for convenience \( A_{n} = A - A_{n-1} \). We will show first that \( \mathcal{L}_{r}^{\kappa} \) is a normed subspace of \( \mathcal{L}^{r} \). Let \( \{A_k\}_{k=1}^{\infty}, \{B_k\}_{k=1}^{\infty} \in \mathcal{L}_{r}^{\kappa} \). Then
\[ \left\| \{ (A_k + B_k)_{n} \}_{k=1}^{\infty} \right\|_r = \left\| \{ (A_k)_{n} \}_{k=1}^{\infty} + \{ (B_k)_{n} \}_{k=1}^{\infty} \right\|_r \leq \left\| \{ (A_k)_{n} \}_{k=1}^{\infty} \right\|_r + \left\| \{ (B_k)_{n} \}_{k=1}^{\infty} \right\|_r \rightarrow 0. \]
Thus \( \mathcal{L}_{r}^{\kappa} \) is closed under addition. It clear that \( \lambda \{A_k\}_{k=1}^{\infty} \in \mathcal{L}_{r}^{\kappa} \) for any complex number \( \lambda \). Hence \( \mathcal{L}_{r}^{\kappa} \) is a normed subspace of \( \mathcal{L}^{r} \). To show that \( \mathcal{L}_{r}^{\kappa} \) is a Banach space, it suffices to show that \( \mathcal{L}_{r}^{\kappa} \) is a closed subspace of \( \mathcal{L}^{r} \). Suppose that
\( \{ A_n = \{ A_k^{(n)} \}_{k=1}^\infty \}_{n=1}^\infty \) is a sequence in \( \mathcal{L}_r^\ast \) converging to an element \( A = \{ A_k \}_{k=1}^\infty \) in \( \mathcal{L}_r^\ast \), and let \( \epsilon > 0 \) be given. Then there is a positive integer \( N \) such that

\[
\| A_N - A \|_r < \frac{\epsilon}{2}.
\]

Due to the membership of \( A_N \) in \( \mathcal{L}_r^\ast \), we have that there exists a positive integer \( J_0 \) such that

\[
\left\| \left\{ (A_k^{(N)})_{\cdot,j} \right\}_{j=1}^\infty \right\|_r < \frac{\epsilon}{2} \text{ for all } J \geq J_0.
\]

It follows that

\[
\left\| \left\{ (A_k^{(N)})_{\cdot,j} \right\}_{j=1}^\infty - \left\{ (A_k)_{\cdot,j} \right\}_{j=1}^\infty \right\|_r + \left\| \left\{ (A_k^{(N)})_{\cdot,j} \right\}_{j=1}^\infty \right\|_r \leq \| A_N - A \|_r + \left\| \left\{ (A_k^{(N)})_{\cdot,j} \right\}_{j=1}^\infty \right\|_r < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for all } J \geq J_0.
\]

Consequently, \( \{ A_k \}_{k=1}^\infty \) belongs to \( \mathcal{L}_r^\ast \).

The following is the main theorem in this thesis. It tells us that the annihilator \( (\mathcal{L}_r^\ast)^\perp \) of \( \mathcal{L}_r^\ast \) is complemented in \( (\mathcal{L}_r^\ast)^\ast \). Furthermore, the norm of the decomposition of any bounded linear functional on \( \mathcal{L}_r^\ast \) is additive.

**Theorem 3.2.5.** Let \( 1 \leq r < \infty \). Then the following hold.

1. The annihilator \( (\mathcal{L}_r^\ast)^\perp \) of \( \mathcal{L}_r^\ast \) is a non-trivial closed subspace of the dual \( (\mathcal{L}_r^\ast)^\ast \) of \( \mathcal{L}_r^\ast \).

2. There is a subspace \( \mathcal{P} \) of \( (\mathcal{L}_r^\ast)^\ast \) such that \( \mathcal{P} \) is isometrically isomorphic to \( (\mathcal{L}_r^\ast)^\ast \) and \( (\mathcal{L}_r^\ast)^\ast = \mathcal{P} \oplus (\mathcal{L}_r^\ast)^\perp \).

3. For any \( f \in (\mathcal{L}_r^\ast)^\ast \), the decomposition \( f = g + h \), where \( g \in \mathcal{P} \) and \( h \in (\mathcal{L}_r^\ast)^\perp \), satisfies \( \| f \| = \| g \| + \| h \| \).

**Proof.** (1) By the Hahn-Banach extension theorem, we have in general that if \( A \) is a non-trivial closed subspace of a Banach space \( X \), then the annihilator \( A^\perp \) of \( A \) is a non-trivial closed subspace of the dual \( X^\ast \) of \( X \). Thus, by this fact, the assertion (1) holds.

(2) Let \( \varphi \in (\mathcal{L}_r^\ast)^\ast \). For each \( k \), let \( \varphi_k : S_{2,2}(\mathbb{C}) \to \mathbb{C} \) be defined by \( \varphi_k(A) = \varphi(s(A; k)) \) for all \( A \in S_{2,2}(\mathbb{C}) \). It is easy to see that \( \varphi_k \) is linear and \( \| \varphi_k \| \leq \| \varphi \| \) for all \( k \). Hence, for each \( k \), the map \( \varphi_k \) belongs to \( (S_{2,2}(\mathbb{C}))^\ast \). Next, let \( B_k^{(\varphi)} = [\varphi_k(E(1; (j,i)))] \) for all \( k \), and let \( B^{(\varphi)} = \{ B_k^{(\varphi)} \}_{k=1}^\infty \). We want to show first that \( \sum\limits_{k=1}^\infty \sum\limits_{i,j \geq 1} (A_k \cdot B_k^{(\varphi)})^{[1]} < \infty \) for all \( \{ A_k \}_{k=1}^\infty \in \mathcal{L}_r^\ast \). To see this,
let \( \mathbf{A} = \left\{ A_k = \left[ a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty} \subset \mathcal{L}_r \). Notice that \( \mathbf{A} \cdot \mathbf{B}(\varphi) \) belongs to \( \mathcal{L}_r \) due to the fact that

\[
\sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \varphi_k(E(1; (j, i))) \right|^{r} \leq \| \varphi \|^{r} \sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^{r}
\]

for all \( j, i \). For each \( k \), let

\[
\tilde{A}_k = \left[ \left( \text{sgn} \left( \varphi_k \left( E \left( a_{ji}^{(k)}; (j, i) \right) \right) \right) \right) a_{ji}^{(k)} \right], \text{ and let } \tilde{\mathbf{A}} = \{ \tilde{A}_k \}_{k=1}^{\infty}.
\]

Then \( \tilde{\mathbf{A}} \in \mathcal{L}_r \) with the same norm as \( \mathbf{A} \). Let \( \nu, \mu \) and \( K \) be positive integers, and let \( n = \max\{\nu, \mu\} \). Then

\[
\sum_{k=1}^{K} \sum_{j=1}^{\nu} \sum_{i=1}^{\mu} \left| a_{ji}^{(k)} \varphi_k(E(1; (j, i))) \right| < \sum_{k=1}^{K} \sum_{j=1}^{n} \sum_{i=1}^{n} \left| \varphi_k \left( E \left( a_{ji}^{(k)}; (j, i) \right) \right) \right|.
\]

\[
= \sum_{k=1}^{K} \sum_{j=1}^{n} \sum_{i=1}^{n} \left( \text{sgn} \left( \varphi_k \left( E \left( a_{ji}^{(k)}; (j, i) \right) \right) \right) \right) \varphi_k \left( E \left( a_{ji}^{(k)}; (j, i) \right) \right)
\]

\[
= \sum_{k=1}^{K} \sum_{j=1}^{n} \sum_{i=1}^{n} \varphi_k \left( E \left( \left( \text{sgn} \left( \varphi_k \left( E \left( a_{ji}^{(k)}; (j, i) \right) \right) \right) \right) a_{ji}^{(k)}; (j, i) \right) \right)
\]

\[
= \sum_{k=1}^{K} \varphi_k \left( \left( \tilde{A}_k \right)_{n_j} \right) = \sum_{k=1}^{K} \varphi \left( \left( \tilde{A}_k \right)_{n_j} ; k \right)
\]

\[
= \varphi \left( \sum_{k=1}^{K} \left( \tilde{A}_k \right)_{n_j} ; k \right) = \varphi \left( \left( \tilde{\mathbf{A}} \right)_{n_j} ; K \right)
\]

\[
\leq \| \varphi \| \left\| \left( \tilde{\mathbf{A}} \right)_{n_j} \right\|_r \leq \| \varphi \| \left\| \tilde{\mathbf{A}} \right\|_r = \| \varphi \| \left\| \mathbf{A} \right\|_r.
\]

It follows that

\[
\sum_{k=1}^{\infty} \sum_{j=1}^{\nu} \sum_{i=1}^{\mu} \left| a_{ji}^{(k)} \varphi_k(E(1; (j, i))) \right| \leq \| \varphi \| \left\| \mathbf{A} \right\|_r.
\]

From this result, we can define a bounded linear functional \( \psi_{\varphi} \) on \( \mathcal{L}_r \) by \( \{ A_k \}_{k=1}^{\infty} \mapsto \sum_{k=1}^{\infty} A_k \cdot B_k(\varphi) \) with \( \| \psi_{\varphi} \| \leq \| \varphi \| \). Notice that for any \( \mathbf{A} = \{ A_k \}_{k=1}^{\infty} \in \mathcal{L}_r \) and positive integer \( K \), we have by the absolute convergence of the series \( \sum A_k \cdot B_k(\varphi) \) \((k = 1, 2, ..., K)\) that

\[
\psi_{\varphi} \left( \left( \mathbf{A}_K \right)_{n_j} \right) = \lim_{n \to \infty} \sum_{k=1}^{K} A_k \cdot B_k(\varphi) = \sum_{k=1}^{K} \lim_{n \to \infty} \sum_{k=1}^{K} \varphi_k((A_k)_{n_j})
\]

\[
= \lim_{n \to \infty} \sum_{k=1}^{K} \varphi(s((A_k)_{n_j}; k)) = \lim_{n \to \infty} \varphi \left( \sum_{k=1}^{K} s((A_k)_{n_j}; k) \right)
\]

\[
= \lim_{n \to \infty} \varphi \left( (\mathbf{A}_{n_j})_K \right) = \lim_{n \to \infty} \varphi \left( (\mathbf{A}_K)_{n_j} \right). \quad (8)
\]
Next, let $\rho_\varphi = \varphi - \psi_\varphi$. We will show that $\rho_\varphi \in \mathcal{L}_r$\textsuperscript{−}. To see this, let $A \in \mathcal{L}_r$. Then $A_n \in \mathcal{L}_r$, and thus $\| (A_n)_{n=1}^\infty - A \|_r \to 0$ for all $K$. Whence, by (8) and the continuity of $\varphi$, we get $\psi_\varphi (A_n) = \lim_{K \to \infty} \varphi (A_n) = \varphi (A)$ for all $K$. Since by Corollary 3.2.3, we have $\| A_n - A \|_r \to 0$, it follows from the continuity of $\psi_\varphi$ and $\varphi$ that $\psi_\varphi (A) = \lim_{K \to \infty} \psi_\varphi (A_n) = \lim_{K \to \infty} \varphi (A_n) = \varphi (A)$, which implies that $\varphi \in \mathcal{L}_r$. Put $\mathcal{P} = \{ \psi_\varphi : \varphi \in \mathcal{L}_r \}$. We claim that $(\mathcal{L}_r)^* = \mathcal{P} \oplus (\mathcal{L}_r)^{\perp}$ and $\mathcal{P}$ is isometrically isomorphic to $(\mathcal{L}_r)^*$. From the definition of $\mathcal{P}$, we have already had that $(\mathcal{L}_r)^* = \mathcal{P} \oplus (\mathcal{L}_r)^{\perp}$. The decomposition $(\mathcal{L}_r)^* = \mathcal{P} \oplus (\mathcal{L}_r)^{\perp}$ will be obtained once it can be shown that $\mathcal{P} \cap (\mathcal{L}_r)^{\perp} = 0$. To see this, let $\psi_\varphi \in \mathcal{P} \cap (\mathcal{L}_r)^{\perp}$ for some $\varphi \in (\mathcal{L}_r)^*$. Then for every $A = \{ A_k \}_{k=1}^\infty \in \mathcal{L}_r$, we have by the absolute convergence of the series

$$\sum_{k=1}^\infty \sum_{k=1}^\infty A_k \cdot B_k^{(\varphi)}$$

and the fact that the sequence $A_n \in \mathcal{L}_r$ for all $n$ that $\psi_\varphi (A) = \lim_{n \to \infty} \psi_\varphi (A_n) = 0$. Therefore, $\psi_\varphi = 0$, which yields $\mathcal{P} \cap (\mathcal{L}_r)^{\perp} = \{ 0 \}$. Accordingly, we have $(\mathcal{L}_r)^* = \mathcal{P} \oplus (\mathcal{L}_r)^{\perp}$ as asserted. The rest is to prove that $\mathcal{P}$ is isometrically isomorphic to $(\mathcal{L}_r)^*$. To get this, we need to show first that $\| \psi_\varphi ||_{\mathcal{L}_r} = \| \psi_\varphi \|$. It is obvious that $\| \psi_\varphi ||_{\mathcal{L}_r} \leq \| \psi_\varphi \|$. To have that $\| \psi_\varphi ||_{\mathcal{L}_r} \geq \| \psi_\varphi \|$, let $\epsilon > 0$ be given. Then there is a sequence $A = \{ A_k \}_{k=1}^\infty \in \mathcal{L}_r$ such that $\| A \|_r \leq 1$ and $\| \psi_\varphi \| < \| \psi_\varphi (A) \| + \epsilon$. Thus, by the absolute convergence of the series $\sum_{k=1}^\infty \sum_{k=1}^\infty A_k \cdot B_k^{(\varphi)}$, there is a positive integer $n$ such that $\| \psi_\varphi \| < \| \psi_\varphi (A_n) \| + \epsilon < \| \psi_\varphi ||_{\mathcal{L}_r} + \epsilon$ for all $\epsilon > 0$. This implies that $\| \psi_\varphi \| \leq \| \psi_\varphi ||_{\mathcal{L}_r}$, and hence we obtain $\| \psi_\varphi ||_{\mathcal{L}_r} = \| \psi_\varphi \|$ as desired. From this result, the map $\psi_\varphi \to \psi_\varphi ||_{\mathcal{L}_r}$ is now an isometric isomorphism from $\mathcal{P}$ into $(\mathcal{L}_r)^*$. To see that it is onto, let $\varphi_0 \in (\mathcal{L}_r)^*$. We then have by the Hahn Banach extension theorem that $\varphi_0$ can extend uniquely to a bounded linear functional $\varphi$ on $\mathcal{L}_r$ with $\| \varphi \| = \| \varphi_0 \|$. Since $\varphi_0$ agrees with $\varphi$ on $\mathcal{L}_r$, it follows $\psi_\varphi ||_{\mathcal{L}_r} = \varphi_0$. Consequently, the map $\psi_\varphi \to \psi_\varphi ||_{\mathcal{L}_r}$ is an isometric isomorphism from $\mathcal{P}$ onto $(\mathcal{L}_r)^*$.

(3). Let $\varphi = \psi_\varphi + \rho_\varphi \in (\mathcal{L}_r)^*$. It is apparent that $\| \varphi \| \leq \| \psi_\varphi \| + \| \rho_\varphi \|$. We want to show that the reverse inequality holds. To prove this, let $\epsilon > 0$ be given. Then there is a sequence $A = \{ A_k \}_{k=1}^\infty \in \mathcal{L}_r$ with $\| A \|_r \leq 1$ such that $\| \psi_\varphi (A) \| > \| \psi_\varphi \| - \frac{\epsilon}{3}$. From this, we have by the absolute convergence of the series $\sum_{k=1}^\infty \sum_{k=1}^\infty A_k \cdot B_k^{(\varphi)}$ that there exists a positive integer $N$ such that $\| \psi_\varphi (A_N) \| > \| \psi_\varphi \| - \frac{\epsilon}{3}$. Let $C = (\text{sgn} \psi_\varphi (A_N)) A_N$. Then $\| C ||_r = \| A_N ||_r \leq \| A \|_r \leq 1$ and $\psi_\varphi (C) = \| \psi_\varphi (A_N) \| > \| \psi_\varphi \| - \frac{\epsilon}{3}$. Next, let $D = \{ D_k \}_{k=1}^\infty \in \mathcal{L}_r$ be such that $\| D \|_r \leq 1$, $\rho_\varphi (D) > 0$ and $\rho_\varphi (D) > \| \rho_\varphi \| - \frac{\epsilon}{3}$. Then by the absolute convergence of the series $\sum_{k=1}^\infty \sum_{k=1}^\infty D_k \cdot B_k^{(\varphi)}$, we have $\psi_\varphi (D_n) \to 0$. Whence there is a positive integer $J > N$ such that $\| \psi_\varphi (D_J) \| < \frac{\epsilon}{3}$. Since $D - D_J \in \mathcal{L}_r$, it follows that $\rho_\varphi (D_J) = \rho_\varphi (D)$. Let $E = C + D_J$. Then
$E \in \mathcal{L}^r$ and by Theorem 1.1(4), we have $\|E\|_r = \max\{\|C\|_r, \|D_{\lambda}\|_r\} \leq 1$. Thus

$$
\|\varphi\| \geq |\psi_{\varphi}(E)| = |\psi_{\varphi}(C) + \psi_{\varphi}(D_{\lambda}) + \rho_{\varphi}(C) + \rho_{\varphi}(D_{\lambda})| \\
= |\psi_{\varphi}(C) + \psi_{\varphi}(D) + \rho_{\varphi}(D)| \\
\geq \psi_{\varphi}(C) + \rho_{\varphi}(D) - |\psi_{\varphi}(D_{\lambda})| \\
> \|\psi_{\varphi}\| - \frac{\epsilon}{3} + \|\rho_{\varphi}\| - \frac{\epsilon}{3} - \frac{\epsilon}{3} \\
= \|\psi_{\varphi}\| + \|\rho_{\varphi}\| - \epsilon.
$$

Since $\epsilon > 0$ was given arbitrarily, it follows that $\|\psi_{\varphi}\| + \|\rho_{\varphi}\| \leq \|\varphi\|$. Hence the equation $\|\varphi\| = \|\psi_{\varphi}\| + \|\rho_{\varphi}\|$ is obtained.

**Remark 3.2.6.** Since $(\mathcal{L}^r_\kappa)^*$ is isometrically isomorphic to $\mathcal{P}$, we may treat $(\mathcal{L}^r_\kappa)^*$ as a subspace of $(\mathcal{L}^r)^*$. Thus Theorem 3.2.5 can be symbolized analogously to Dixmier’s theorem as follows:

$$(\mathcal{L}^r)^* = (\mathcal{L}^r_\kappa)^* \oplus_1 (\mathcal{L}^r_\kappa)_s.$$
Chapter 4

Conclusion

Let $\mathcal{M}_\infty$ be the set of all infinite complex matrices. For each $1 \leq r < \infty$, we define a class of sequences of infinite complex matrices $\mathcal{L}^r$ as follows:

$$\mathcal{L}^r = \left\{ \left\{ a^{(k)}_{ji} \right\}_{k=1}^{\infty} \subseteq \mathcal{M}_\infty : \left[ \sum_{k=1}^{\infty} \left| a^{(k)}_{ji} \right|^r \right] \in \mathcal{B}(l^2) \right\},$$

and for any sequence $\left\{ a^{(k)}_{ji} \right\}_{k=1}^{\infty}$ in $\mathcal{M}_\infty$, we define

$$\left\| \left\{ a^{(k)}_{ji} \right\}_{k=1}^{\infty} \right\|_r = \left\{ \left[ \sum_{k=1}^{\infty} \left| a^{(k)}_{ji} \right|^r \right] \right\}^{1/r} \text{ if } \left\{ a^{(k)}_{ji} \right\}_{k=1}^{\infty} \in \mathcal{L}^r,$$

otherwise.

In this thesis, we study some elementary properties and provide some results on the duality of $\mathcal{L}^r$. The main goal is to decompose the dual space $(\mathcal{L}^r)^*$ of $\mathcal{L}^r$ as an $l^1$ direct-sum of its two closed subspaces by a way analogous to a beautiful theorem of Dixmier on decomposing the dual $\mathcal{B}(l^2)^*$ of $\mathcal{B}(l^2)$. The following are the results.

We first obtain some characterizations of the sets $\mathcal{L}^r$.

**Theorem 4.1.** Let $\left\{ A_k \right\}_{k=1}^{\infty}$ be a sequence in $\mathcal{M}_\infty$ and $1 \leq r < \infty$. Then the following are equivalent:

1. $\left\{ A_k \right\}_{k=1}^{\infty}$ belongs to $\mathcal{L}^r$;
2. $A_k \in S_{2,2}^r(\mathbb{C})$ for all $k$ and the sequence $\left\{ \sum_{k=1}^{n} A_k^{[r]} \right\}_{n=1}^{\infty}$ is bounded in $\mathcal{B}(l^2)$;
3. the sequence $\left\| \sum_{k=1}^{n} A_k^{[r]} \right\|_{n=1}^{\infty}$ is bounded.

To obtain that $\| \cdot \|_r$ is precisely a norm on $\mathcal{L}^r$, the following Hölder-type inequality is constructed.
Theorem 4.2. (Hölder-type inequality for sequences of matrices) For any sequences \( \{A_k\}_{k=1}^{\infty} \) and \( \{B_k\}_{k=1}^{\infty} \) in \( M_\infty \),
\[
\| \{ A_k \cdot B_k \}_{k=1}^{\infty} \|_1 \leq \| \{ A_k \}_{k=1}^{\infty} \|_r \cdot \| \{ B_k \}_{k=1}^{\infty} \|_{r^*},
\]
where \( 1 < r < \infty \) with \( \frac{1}{r} + \frac{1}{r^*} = 1 \), under the conventions that \( \infty \cdot 0 = 0 \cdot \infty = 0 \), \( \infty \cdot \alpha = \alpha \cdot \infty = \infty \) for all positive real number \( \alpha \) and \( \infty + \infty = \infty \).

From the Hölder-type inequality, the corresponding Minkowski’s inequality is obtained.

Theorem 4.3. (Minkowski-type inequality for sequences of matrices) For any sequences \( \{A_k\}_{k=1}^{\infty} \) and \( \{B_k\}_{k=1}^{\infty} \) in \( M_\infty \) and \( 1 \leq r < \infty \),
\[
\| \{ A_k + B_k \}_{k=1}^{\infty} \|_r \leq \| \{ A_k \}_{k=1}^{\infty} \|_r + \| \{ B_k \}_{k=1}^{\infty} \|_{r^*},
\]
under the conventions that \( \infty + \alpha = \alpha + \infty = \infty \) for all non-negative real number \( \alpha \) and \( \infty + \infty = \infty \).

From the Minkowski’s inequality, we have that the set \( L^r \) equipped with the norm \( \| \|_r \) is a normed space. A Riez-fischer-type theorem for completeness of the sequence spaces \( L^r \) is obtained below.

Theorem 4.4. For each \( 1 \leq r < \infty \), the set \( L^r \) equipped with the norm \( \| \|_r \) is a Banach space.

For the classical sequence spaces \( l^p \) (\( 1 \leq p < \infty \)), there is a result closely related to their duality stating that for every \( 1 \leq p < \infty \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), a sequence \( \{x_k\}_{k=1}^{\infty} \) belongs to \( l^q \) if and only if \( \{x_k y_k\}_{k=1}^{\infty} \in l^1 \) for all \( \{y_k\} \in l^p \). We obtain a similar duality-type result for the sequence spaces \( L^r \) as follows.

Theorem 4.5. Let \( 1 < r < \infty \) with \( \frac{1}{r} + \frac{1}{r^*} = 1 \).

(1) A sequence \( \{A_k\}_{k=1}^{\infty} \in L^{r^*} \) if and only if \( \{A_k \cdot B_k\}_{k=1}^{\infty} \in L^1 \) for all \( \{B_k\}_{k=1}^{\infty} \in L^r \).

(2) If \( \{A_k\}_{k=1}^{\infty} \in L^{r^*} \), then
\[
\| \{ A_k \}_{k=1}^{\infty} \|_{r^*} = \sup \{ \| \{ A_k \cdot B_k \}_{k=1}^{\infty} \|_1 : \{ B_k \}_{k=1}^{\infty} \in L^r, \| \{ B_k \}_{k=1}^{\infty} \|_r \leq 1 \}.
\]

Next, we define the class
\[
L^r_{\kappa} = \left\{ \left\{ a^{(k)}_{ji} \right\}_{k=1}^{\infty} \subset M_\infty : \sum_{k=1}^{\infty} \left| a^{(k)}_{ji} \right|^r \in K(l^2) \right\},
\]
We obtain a characterization of \( L^r_{\kappa} \) as follows.
Theorem 4.6. Let \( \{A_k\}_{k=1}^{\infty} \) be a sequence in \( \mathcal{M}_\infty \) and \( 1 \leq r < \infty \). Then \( \{A_k\}_{k=1}^{\infty} \in \mathcal{L}_r^\kappa \) if and only if \( A_k^{[r]} \) is compact for all \( k \) and the sequence \( \left\{ \sum_{k=1}^{n} A_k^{[r]} \right\}_{k=1}^{\infty} \) converges in \( \mathcal{B}(l^2) \).

Theorem 4.7. For each \( 1 \leq r < \infty \), the set \( \mathcal{L}_r^\kappa \) is a Banach subspace of \( \mathcal{L}_r^r \).

We finally obtain a decomposition theorem for the dual \( \mathcal{L}_r^r \) of \( \mathcal{L}_r^r \) as follows.

Theorem 4.8. Let \( 1 \leq r < \infty \). Then the following hold.

1. The annihilator \( (\mathcal{L}_r^\kappa)^\perp \) of \( \mathcal{L}_r^\kappa \) is a non-trivial closed subspace of the dual \( (\mathcal{L}_r^r)^* \) of \( \mathcal{L}_r^r \).

2. There is a subspace \( \mathcal{P} \) of \( (\mathcal{L}_r^r)^* \) such that \( \mathcal{P} \) is isometrically isomorphic to \( (\mathcal{L}_r^\kappa)^* \) and \( (\mathcal{L}_r^r)^* = \mathcal{P} \oplus (\mathcal{L}_r^\kappa)^\perp \).

3. For any \( f \in (\mathcal{L}_r^r)^* \), the decomposition \( f = g + h \), where \( g \in \mathcal{P} \) and \( h \in (\mathcal{L}_r^\kappa)^\perp \), satisfies \( \|f\| = \|g\| + \|h\| \).

Notice that Theorem 4.8 can be symbolized analogously to Dixmier’s theorem as follows:

\[
(\mathcal{L}_r^r)^* = (\mathcal{L}_r^\kappa)^* \oplus (\mathcal{L}_r^\kappa)_s.
\]
Bibliography


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